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# Semiclassical quantum dynamics via the method of Stationary Phase for a rigorous approach to Feynman Path Integrals

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# Abstract

The Feynman path integral is a heuristic tool widely used by physicists from many different branches. The probably best known application is the calculation of probability amplitudes in quantum theory. Since its introduction by Richard P. Feynman in 1948, as an alternative approach to non-relativistic quantum mechanics, many attempts have been made to put the theory on a rigorous mathematical footing, each with its own advantages and shortcomings.

In this thesis, we will follow the treatment due to Albeverio and Høegh-Krohn, which is based on Hörmander's finite-dimensional theory of oscillatory integrals, and therefore relies directly on the oscillatory nature of the path integral.

By restricting the class of potentials to Fourier transforms of complex measures, solutions to the Schrödinger equation in the form of infinite-dimensional oscillatory integrals are given. As an application of the theory's own method of stationary phase, it follows that in the semiclassical regime, the main contribution to the probability amplitude is determined by the classical path.

# Acknowledgements

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# Contents

# 1 Introduction

The main purpose of this thesis is a mathematical treatment of the Feynman path integral together with an appropriate method of sationary phase, in order to obtain a semiclassical approximation of solutions to the Schrödinger equation for a non-relativistic particle experiencing an external potential.

Starting from the well-known theory of finite-dimensional oscillatory integrals, in chapter 2 we will pass to a construction of oscillatory integrals in infinite dimensions. By applying the theory to quantum mechanics in section 2.4, the developed theory will be used to construct solutions to the Schrödinger equation, providing a rigorously defined path integral formula. In chapter 3, a general stationary phase approximation for oscillatory integrals is developed, which will be used in section 3.2 to study the behaviour of the path integral solutions of chapter 2 in the semiclassical regime.

**The heuristic Feynman path integral.** In his Ph.D. thesis from 1948 [13], the physicist Richard P. Feynman came up with a new heuristic approach to non-relativistic quantum dynamics, known as the *Feynman path integral formulation of quantum mechanics*. As can be demonstrated, for example by the famous Double Slit experiment, a quantum system always knows of all possible ways to pass between two states, in the sense that the total probability to pass from one quantum state to another depends on the probabilities of all possible *paths* connecting these two states. Based on this idea, Feynman postulated a formula for the total probability amplitude in the form of a weighted sum over all possible virtual paths, generally known as a *sum over histories*, where the contribution of each path is determined from the value of the classical action for the given path.

From a mathematical point of view, the problem arises as soon as one considers systems with infinite degrees of freedom, where the sum over histories turns out to be an integral on an infinite dimensional space of paths. In the case of a non-relativistic point particle moving in the field of an exterior potential, a typical path integral formula takes the heuristic form

$$\int_{\Omega} e^{iS(x)}\, \mathcal{D}x$$

where $S(x)$ denotes the classical action along the curve $x$ and $\Omega$ is some subset of $C([0,t],\mathbb{R}^d)$, specified by imposing suitable boundary conditions. Whereas $\mathcal{D}x$ in general has no rigorous meaning in terms of measure theory, but rather has to be determined heuristically in each given situation separately, under the requirement that all paths in the domain shall be weighted uniformly.

Even though widely used by physicists, there are several apparent mathematical difficulties with such an expression, which are often sucessfully neglected in physics literature. First, it is not à priori clear, why $\Omega$ should be contained in the domain of $S$, which consisits of terms involving the derivative of the paths, as well as the potential.

Next, even in the simple case when $\Omega$ is one-dimensional, for example when it only consists of linear functions on $[0,t]$, indexed by their slopes $\xi \in \mathbb{R}^d$, i.e. $x_\xi(s) = s\xi$, it is still questionable in which sense the expression $\int e^{iS(x_\xi)} d\xi$ exists, since $S$ is real-valued and therefore $e^{iS}$ is not Lebesgue integrable. However, in the general case when $\Omega$ is infinite-dimensional, the most obvious difficulty is to give meaning to the heuristic expression $\mathcal{D}x$.

**Rigorous approaches.** Despite all its mathematical shortcomings, most physicists consider the Feynman path integral as a valuable tool, which is why many attempts have been made to provide a suitable mathematical framework. But as of today, there is no rigorous theory which is general enough to cover all physically relevant cases of potentials, and at the same time provides new mathematical content instead of merely being a reformulation of the canonical approach.

One of the early attempts to rigorously define path integrals was by analytic continuation of Wiener integrals, first introduced by Cameron [8], and further developed by many other authors (e.g. [24]), but a stationary phase approximation and at many points rigorous results are missing.

Next, there is the sequential or time-slicing approach, where the initial idea already came from Feynman himself. By using the Lie-Trotter product formula on the unitary group generated by the Hamilton operator of the system, together with a suitable partition of the time interval, one can write the unitary group as a strong limit of finite-dimensional integral operators, which take the form of discretized path integrals. A detailed exposition together with connections to the analytic treatment can be found in [23].

Moreover, there are two separate probabilistic approaches, one that is more practically oriented and connected with Poisson processes [25], and another one based on Hida distribution theory [18], which is more abstract by nature.

However, in this thesis, we will follow a completely different treatment, which directly focusses on the oscillatory nature of path integrals, and is based on the well-known finite-dimensional theory of oscillatory integrals by Hörmander [21, 7.8]. However, the major flaw of the theory is its limited applicability in physics, because it only works for potentials that are Fourier transforms of complex measures and satisfy certain regularity conditions (see the explanation below, and Theorem 3 for more details). It can however by slightly extended to cover the case of an additional harmonic oscillator term in the potential (see [1]), as well as polynomially growing terms [6].

**Detailed structure of the thesis.** Chapter 2 is devoted to the introduction of oscillatory integrals and their application to quantum mechanics, leading to solutions of the Schrödinger equation for a non-relativistic point particle in the form of a Feynman path integral. We start with the definition of oscillatory integrals on $\mathbb{R}^n$ in section 2.1, followed by some simple properties, which will be needed later.

In 2.2, after a short survey on complex measures in a general measurable space, we introduce the Fresnel class $\mathscr{F}(\mathcal{H})$, the space of Fourier transforms of complex measures on a real separable Hilbert space $\mathcal{H}$. The section is closed by Theorem 1, which shows that for any $f{=}\hat{\mu} \in \mathscr{F}(\mathbb{R}^n)$ its finite-dimensional oscillatory integral exists and can be calculated by an explicit expression involving an ordinary integral with respect to the measure $\mu$ itself. This result is referred to as Cameron-Martin type formula, due to its similarity to the original Cameron-Martin formula for Wiener integrals [9].

Section 2.3 consists of two parts: First, we show how oscillatory integrals on $\mathbb{R}^n$ can be extended to arbitrary finite-dimensional inner product spaces by using any basis representation, and a Cameron-Martin type formula equivalent to Theorom 1 is established in Proposition 2. After that, the extension to infinite-dimensional separable Hilbert spaces, in form of the *normalized Fresnel integral*, is made and the final Cameron-Martin type theorem is presented in Theorem 2.

In section 2.4, the so called Cameron-Martin space $\mathcal{H}_t$ is introduced, which consists of all $x \in H^1(0,t;\mathbb{R}^d)$ with the property $x(t){=}0$, equipped with the inner product $\langle x,y\rangle_t := \langle x',y'\rangle_{L^2([0,t])}$. Theorem 3 then states the path integral formula based on the normalized Fresnel integral on $\mathcal{H}_t$, which will be proved in the rest of the section by first showing several intermediate results (Propositions 4-9, Lemmas 4-6) that are then put together in the proof starting on page 45. An explanation of the statement can be found below.

Chapter 3 contains two main results, Theorem 4 and 5, which describe a stationary phase approximation of Fresnel and path integrals, respectively. In section 3.1, after some introductory results and definitions, Theorem 4 states the main ingredient for the stationary phase approximation of the path integral from Theorem 3. Under certain regularity conditions, we give an approximation of a special class of normalized Fresnel integrals depending on a parameter $h$.

This result will then be used in Section 3.2, in order to give an approximation of path integral solutions to the macroscopic Schrödinger equation in the semiclassical regime.

**Explanation of the main results.** Theorem 3 shows that strong solutions to the Schrödinger equation $\frac{d}{dt}\psi_t = -iH\psi_t$, for $H = H_0{+}V$, $H_0 = -\Delta/2$, in $d$ dimensions, can be constructed by using the normalized Fresnel integral from Chapter 2. The crutial assumptions are, that $V$ as well as the initial datum $\psi_0{=}\varphi$ are Fourier transforms of complex measures, while $V$ has to be at least two times continously differentiable with bounded first and second order derivatives, and $\varphi$ has to belong to $\mathcal{D}(H) = \mathcal{D}(H_0) = H^2(\mathbb{R}^d)$, the Sobolev space of second order, on which $H$ is self-adjoint due to Kato-Rellich. Under these hypotheses, we prove that $\psi_t$, defined by

$$\psi_t(\xi) \;:=\; \mathcal{F}_t\Big(x \mapsto e^{-i\int_0^t V(x(s)+\xi)ds}\,\varphi(x(0)+\xi)\Big)$$

is in $H^2(\mathbb{R}^d)$ and solves the Schrödinger equation, where $\mathcal{F}_t(f)$ denotes the normalized Fresnel integral of a function $f$ on the Cameron-Martin space $\mathcal{H}_t$. The precise definition of $\mathcal{F}_t(f)$ can be found in Definition 4, but here let us informally explain, how it is constructed. First one defines oscillatory integrals in finite dimensions, usually denoted by $\int^o e^{\frac{i}{2}|x|^2} g(x)dx$, as the limit of $\int e^{\frac{i}{2}|x|^2} g(x)\psi(\varepsilon x)dx$ as $\varepsilon \to 0$, where $\psi$ denotes a Schwartz function such that $\psi(0) = 1$. If the limit exists and does not depend on $\psi$, then $g$ is called Fresnel integrable.

For example, from the Cameron-Martin type formulas (Theorem 1, Proposition 2), we obtain $\int^o e^{\frac{i}{2}|x|^2} dx = (2\pi i)^{n/2}$, which, in absolute value, grows exponentially with the dimension $n$ of the underlying space. Hence, before passing to infinite dimensions, one wants to normalize the Fresnel integral properly (by multiplying with $(2\pi i)^{-n/2}$).

The extension to general, perhaps infinite-dimensional, separable Hilbert spaces $\mathcal{H}$, is then done by means of sequences of finite-dimensional projections $(\mathcal{P}_n)_{n\in\mathbb{N}}$. More precisely, the normalized Fresnel integral on $\mathcal{H}$, denoted by $\mathcal{F}_{\mathcal{H}}(f)$ or also by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\|x\|^2} f(x)\, dx$$

is defined as the limit of the sequence of normalized Fresnel integrals on $\mathcal{P}_n\mathcal{H}$, whenever the limit exists and is independent of $(\mathcal{P}_n)_{n\in\mathbb{N}}$. For example, on the Cameron-Martin space $\mathcal{H}_t$, in the proof of Theorem 3, the $\mathcal{P}_n$ are taken to be projections from $\mathcal{H}_t$ to piecewise linear functions on $[0, t]$ (see Examples 2 and 3).

Recalling, that in our case the classical action along a path $y \in H^1(0, t; \mathbb{R}^d)$ is given by $S(y) := \int_0^t \big(|y'(s)|^2/2 - V(y(s))\big)ds$, for the Fresnel integral $\psi_t(\xi)$ we may informally write

$$\psi_t(\xi) = \widetilde{\int_{\mathcal{H}_t}} e^{iS(x+\xi)}\, \varphi(x(0) + \xi)\, dx$$

where $x + \xi$ denotes the path $s \mapsto x(s) + \xi$, with endpoint $\xi \in \mathbb{R}^d$, which takes the form of a path integral as introduced by Feynman in [13].

The other core result is given in Theorem 5, which is a direct application of Theorem 4 and gives a semiclassical approximation of the path integral solution obtained in Theorem 3. It says that, if the potential $V$ and initial datum $\varphi$ are Fourier transforms of complex measures, which have finite moments of all orders (this translates into high regularity of $V$ and $\varphi$), from the general stationary phase approximation for Fresnel integrals developed in Theorem 4, at lowest order in the semiclassical parameter $h$, the path integral solution to the macroscopic Schrödinger equation, $\psi_t^h(\xi)$, is determined by the classical path $\gamma$, with $\gamma(t) = \xi$ and $\dot\gamma(0) = 0$.

The precise result of Theorem 4 is in direct accordance with [26, Theorem 12.5], which is obtained from the standard approach to semiclassical analysis, and it can be shown that it leads to the physical result that the semiclassical wave function concentrates around its classical trajectory.

**Remark on notation.** Many authors like to stick with fixed letters for configuration and Fourier space variables, e.g. $x$ and $k$ respectively, but since in our setting, most functions living on the configuration space are Fourier transforms of complex measures, we won't use any convention of this type.

# 2 Oscillatory integrals

When speaking of *oscillatory integrals*, we mean expressions of the form

$$\int_{\Gamma} e^{i\phi(x)} f(x)\, dx \tag{2.1}$$

where the *phase function* $\phi$ is taken to be real-valued and $f$ is a complex *amplitude function*, both being defined on a certain domain $\Gamma$, which together with $\phi$ and $f$ has to be chosen in such a way that the expression makes sense either as an ordinary Lebesgue or Riemann integral or as a properly defined extension of these. Dealing with such extensions is the main concern of this section.

Intuitively we can understand why asking for absolute integrability, for example in the case $\Gamma = \mathbb{R}$, could be too rough: By taking absolute values in (2.1), the oscillating behaviour of $e^{i\phi}$ would be completely suppressed, whereas in regions where $\phi$ varries strongly in comparison to $f$, the positive and negative parts of the oscillating integrand will nearly cancel out, so that these contributions will remain small. Consequently the integral is likely to be finite for more general amplitudes than in the non-oscillating case. This is the idea that we should have in mind, when we rigorously justify the existence of oscillatory integrals in what follows.

## 2.1 Hörmander's finite-dimensional oscillatory integral

We begin with the standard definition of oscillatory integrals in finitely many dimensions due to L. Hörmander [20], which was largely developed as one of the basic tools for studying pseudo-differential and Fourier integral operators with applications in the framework of partial differential equations [21]. We consider the case $\Gamma = \mathbb{R}^n$ in expression (2.1) for some $n \in \mathbb{N}$, i.e. integrals of the form

$$I^{\phi}(f) := \int_{\mathbb{R}^n} e^{i\phi(x)} f(x)\, dx \tag{2.2}$$

In this case, if $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$, then $I^{\phi}(f)$ can be viewed as an ordinary Lebesgue integral only for amplitudes $f$ in $L^1(\mathbb{R}^n, dx)$. But it is one of the most striking features of (2.2), that there is a strictly bigger class of amplitudes to which $I^{\phi}$ can be extended, while some of the most useful properties of ordinary integrals can be kept valid or translated into appropriate versions for the given extension.

In [20, 21] finite dimensional oscillatory integrals for quite general phase functions are defined[1]. However for the concerns of this work, it will suffice to consider phase

[1]More precisely, in [21] $\phi$ is taken to be in $C^{\infty}(\Omega)$ with $\Omega$ being a certain subset of $\mathbb{R}^n$ s.th. $\phi$ is homogeneous of order 1, $Im(\phi) \geqslant 0$ and $\phi$ is not allowed to contain any critical point in $\Omega$.

functions of a much simpler form. In the finite-dimensional case we will deal with phase functions which are proportional to the square of the Euclidean norm in $\mathbb{R}^n$, more precisely we use $\phi = |\cdot|^2/2\rho$, where $\rho > 0$ is some fixed constant. Later, when we apply the theory to quantum mechanics, $\rho$ will be replaced by an expression involving a semiclassical paramter $h$. Such oscillatory integrals—or more generally, when $\phi$ is a non-degenerate real quadratic form—are said to be of *Fresnel type*. The following definition is taken from [32, p. 33] and modified in order to fit into our setting.

**Definition 1** *(Finite-dimensional oscillatory Fresnel integral).* Let $\mathcal{S}^*$ denote the space of Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with the additional property $\varphi(0) = 1$ and let $f$ be a measurable function on $\mathbb{R}^n$, such that the integral

$$I_\varepsilon(\varphi, f) := \int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} f(x)\,\varphi(\varepsilon x)\,dx \tag{2.3}$$

exists for all $\rho > 0$, $\epsilon > 0$ and all $\varphi \in \mathcal{S}^*$. We say that $f$ is *Fresnel integrable*, whenever the limit $\lim_{\varepsilon\to 0^+} I_\varepsilon(\varphi, f)$ exists and is independent of $\varphi \in \mathcal{S}^*$. In this case the limit is denoted by

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(x)\,dx \tag{2.4}$$

and is called the *(finite-dimensional oscillatory) Fresnel integral* of $f$.

Our first task now will be to identify some subclasses of Fresnel integrable functions. First of all, any $f \in L^1(\mathbb{R}^n)$ is Fresnel integrable, because in this case the integrand in (2.3) is dominated by $\|\varphi\|_\infty |f| \in L^1(\mathbb{R}^n)$ and therefore, by using Lebesgue's dominated convergence theorem, the limit can be taken inside of the integral. Due to $\varphi(0) = 1$, this shows that (2.4) coincides with the ordinary Lebesgue integral $\int e^{i|x|^2/2\rho} f(x)\,dx$.

But the theory of oscillatory integrals would not be of much use, if there didn't exist more interesting classes of Fresnel integrable functions. In [20, 21] a detailed treatment of a quite large class of such functions is given, which are called *symbols*. These functions together with their oscillatory integrals are of special importance in the theory of pseudo-differential and Fourier integral operators.

However, in view of later applications we shall focus on another interesting set of Fresnel integrable functions, which consists of Fourier transforms of complex measures. The main advantage in considering these—in the literature also known as functions of *Fresnel class*—is the fact, that in this case oscillatory integrals can be explicitly computed in terms of ordinary integrals with respect to the associated complex measure (see Theorem 1).

When we extend oscillatory integrals to infinite-dimensional separable Hilbert spaces in section 2.3 and apply the theory to quantum mechanics in section 2.4, we will make heavy use of this class as well. Therefore, in the following section we shall study its definition and most important properties in more detail and generality.

Before that, let us proof some first simple properties inherited from ordinary integrals, which we will need later. The reference for Lemmas 1 and 2 is [32].

**Lemma 1** *(Unitary change of variables). If $U$ is a real orthogonal $n \times n$ matrix and $f$ a Fresnel integrable function on $\mathbb{R}^n$, then the function $x \mapsto f(Ux)$ is also Fresnel integrable and it holds*

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(Ux)\, dx \;=\; \int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(x)\, dx \tag{2.5}$$

***Proof.*** Due to $|\det U| = 1$ and $|Ux| = |x|$ for all $x \in \mathbb{R}^n$, performing the change of variables $x \mapsto Ux =: y$ in the integral $\int e^{\frac{i}{2\rho}|x|^2} f(Ux)\,\varphi(\varepsilon x)\,dx$, where $\varphi \in \mathcal{S}^*$ and $\varepsilon > 0$ are arbitrary, gives $\int e^{\frac{i}{2\rho}|y|^2} f(y)\,\varphi(U^T \varepsilon y)\,dy$. Since $y \mapsto \varphi(U^T y)$ is in $\mathcal{S}^*$ and $f$ is Fresnel integrable, by taking the limit $\varepsilon \to 0$, we obtain the Fresnel integral of $f$. □

In view of later applications, it will be usefull to allow phase functions in oscillatory integrals, which are of the more general form $x \mapsto |Ax|^2$, for a real invertible $n \times n$ matrix $A$. But due to the following result, this case can be reduced to the already defined oscillatory Fresnel integral of Definition 1.

**Lemma 2** *(More general phase functions). If $A$ is a real invertible $n \times n$ matrix and if $f$ is a measurable function on $\mathbb{R}^n$, such that $x \mapsto f(A^{-1}x)$ is Fresnel integrable, then for all $\varphi \in \mathcal{S}^*$ and $\rho > 0$, it holds*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|Ax|^2} f(x)\,\varphi(\varepsilon x)\,dx \;=\; \frac{1}{|\det A|} \int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|y|^2} f(A^{-1}y)\,dy \tag{2.6}$$

*Therefore, for those functions $f$, it makes sense to use the notation*

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|Ax|^2} f(x)\,dx \;:=\; \frac{1}{|\det A|} \int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|y|^2} f(A^{-1}y)\,dy \tag{2.7}$$

***Proof.*** The proof consists of integration by substitution on the left-hand side, the fact that $x \mapsto \varphi(A^{-1}x)$ is in $\mathcal{S}^*$, and $x \mapsto f(A^{-1}x)$ being Fresnel integrable. □

## 2.2 The Fresnel class

Let us first review some basic notions and results involving complex measures on a general measurable space $(X, \mathcal{A})$, taken from [31, Chapter 6].

We say that a complex-valued set function on $\mathcal{A}$ is a *complex measure*, if it satisfies $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ for any countable partition of $A \in \mathcal{A}$, that is a collection $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ of pairwise disjoint measurable sets with $A = \cup_{i \in I} A_i$.

In particular, this implies that positive measures are not a subclass of complex measures, since the former are allowed to take infinite values, whereas in the complex case $|\mu(A)|$ has to be finite for all $A \in \mathcal{A}$. The countable additivity then yields, that for any partition $\{A_i\}_{i=1}^{\infty}$ the series $\sum_{i=1}^{\infty} \mu(A_i)$ has to converge in $\mathbb{C}$.

In [31, p. 117] it is shown that a complex measure $\mu$ is dominated by a unique positive measure, defined by $|\mu|(A) := \sup \sum_i |\mu(A_i)|$, where the supremum is taken over all countable partitions $\{A_i\}_{i=1}^{\infty}$ of $A$. It is called the *total variation measure* of $\mu$ and it turns out to be *finite*. The *total variation* of $\mu$ is the number $\|\mu\| := |\mu|(X)$, which defines a norm on $\mathscr{M}(X)$, the linear space of complex measures on $(X, \mathcal{A})$.

A useful property of complex measures involving the total variation measure is the existence of a *polar representation* [31, 6.12]: *For any complex measure $\mu$ on $(X, \mathcal{A})$ there exists a unique complex-valued measurable function $h$ with $|h(x)| = 1$ for all $x \in X$ and*

$$\mu(E) = \int_E h \, d|\mu| \tag{2.8}$$

for all $E \in \mathcal{A}$, i.e. $d\mu = h \, d|\mu|$ for short. Also the converse is true [31, 6.13]: *If $\nu$ is a finite positive measure on $\mathcal{A}$ and $h$ a complex-valued measurable function on $X$ with $|h(x)| = 1$ for all $x \in X$, then defining $\mu(E)$ by (2.8) for all $E \in \mathcal{A}$ gives a complex measure on $\mathcal{A}$ with $|\mu| = \nu$.*

The polar representation can be used to define integration with respect to a complex measure $\mu$ in a straight-forward way [31, p. 129]: Let $h \, d|\mu|$ be the polar representation of $\mu$, then a measurable function $f$ is called $\mu$-integrable, if it is integrable with respect to $|\mu|$ and in this case, one defines $\int f d\mu := \int f h \, d|\mu|$ and $\|f\|_{L^1(X,\mu)} := \int_X |f| \, d|\mu|$.

The *complex product measure* $\mu \times \nu$ of two complex measures $\mu$, $\nu$ is defined in a very similar way. If $d\mu = h \, d|\mu|$ and $d\nu = g \, d|\nu|$ are polar representations, then for any $A \in \mathcal{A} \otimes \mathcal{A}$ one sets

$$\mu \times \nu(A) := \int 1_A(x, y) \, h(x) \, g(y) \, d(|\mu| \times |\nu|)(x, y) \tag{2.9}$$

Since the integrand is bounded (by 1) and $|\mu|, |\nu|$ are finite, we can apply Fubini's theorem in order to get the identity $\mu \times \nu(A \times B) = \mu(A) \, \mu(B)$ for any $A, B \in \mathcal{A}$, which is usually used to construct the product measure for positive measures. Here (2.9) is taken as definition, in order to avoid the complication of having to extend a complex set function on the rectangles to a complex measure on the whole $\sigma$-algebra.

Before we return to the problem of finding Fresnel integrable functions on $\mathbb{R}^n$, we define the so called *Fresnel class* on a general real seperable Hilbert space $\mathcal{H}$.

**Definition 2** *(Fresnel class).* Let $\mathscr{M}(\mathcal{H})$ denote the linear space of complex measures on a real seperable Hilbert space $\mathcal{H}$, equipped with its Borel $\sigma$-algebra. The set of *Fourier transforms* of such measures, i.e. all bounded continuous functions on $\mathcal{H}$ of the form

$$\hat{\mu}(k) := \int_{\mathcal{H}} e^{i\langle k, x\rangle} d\mu(x) \tag{2.10}$$

where $\mu \in \mathscr{M}(\mathcal{H})$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$, is called the *Fresnel class* of $\mathcal{H}$ and denoted by $\mathscr{F}(\mathcal{H})$.

As is true for the Fourier transform on $L^2$-spaces, an element of $\mathscr{F}(\mathcal{H})$ carries the complete information of the underlying measure. In other words, the Fourier transform $\mathscr{M}(\mathcal{H}) \to \mathscr{F}(\mathcal{H})$, $\mu \mapsto \hat{\mu}$ is *one-to-one* [7, 3.8.6].

The property of Fourier transforms of mapping convolution to pointwise multiplication can easily be seen to hold in this setting, too. As a consequence, we obtain that $\mathscr{F}(\mathbb{R}^n)$ is closed under pointwise multiplication. The *convolution* $\mu * \nu$ of complex measures $\mu, \nu \in \mathscr{M}(\mathcal{H})$ is the set function on $\mathcal{B}(\mathcal{H})$ defined by the $\mu \times \nu$-integral of $(x, y) \mapsto \mathbb{1}_A(x+y)$, i.e. $\mu * \nu(A) = \int 1_A(x+y)\, d(\mu \times \nu)(x, y)$ for any $A \in \mathcal{B}(\mathcal{H})$. Since we have $|\mu * \nu(A)| \leqslant |\mu| * |\nu|(A)$ for all $A \in \mathcal{B}(\mathcal{H})$ and $|\mu| * |\nu|$ is finite, it is an application of the theorem of monotone convergence to see that $\mu * \nu$ is a complex measure.

**Proposition 1** *(closedness under multiplication). If $f, g \in \mathscr{F}(\mathcal{H})$, i.e. $f = \hat{\mu}$ and $g = \hat{\nu}$ for some $\mu, \nu \in \mathscr{M}(\mathcal{H})$, then their pointwise product $fg$ is also contained in $\mathscr{F}(\mathcal{H})$ and moreover it holds $fg = \widehat{\mu * \nu}$.*

***Proof.*** If $d\mu = h\, d|\mu|$ and $d\nu = g\, d|\nu|$ are the polar representations of $\mu$ and $\nu$, then for any bounded function $u$ on $\mathcal{H}$, it holds $\int u\, d(\mu * \nu) = \int u(x + y)\, d(\mu \times \nu)(x, y)$, which is equal to $\int (\int u(x + y)\, h(x) d|\mu|(x))\, g(y)\, d|\nu|(y)$, since it is allowed to apply Fubini's theorem due to the bounded integrand and the finiteness of $|\mu|$ and $|\nu|$. Thus, we find for any $k \in \mathcal{H}$: $\widehat{\mu * \nu}(k) = \int e^{i\langle k,x\rangle}\, d\mu(x) \int e^{i\langle k,y\rangle} d\nu(y) = \hat{\mu}(k)\, \hat{\nu}(k) = f(k)g(k)$. □

This implies that $\mathscr{F}(\mathcal{H})$, equipped with pointwise addition and multiplication, forms an algebra, where its unit element is given by the Dirac measure $\delta_0$.

Even though, we don't need it in this thesis, let us remark, that we have even more than that: The property $\|\mu * \nu\| = |\mu \times \nu|(\mathcal{H}) \leqslant |\mu|(\mathcal{H})\, |\nu|(\mathcal{H}) = \|\mu\|\, \|\nu\|$ of complex measures $\mu$, $\nu$ makes $\mathscr{M}(\mathcal{H})$ to a normed algebra with respect to convolution and, as is shown in [7, 4.6.1], it is complete with respect to the total variation norm, hence forming a Banach algebra.

Uniqueness of the Fourier transform together with Proposition 1 allow us to carry these properties over to $\mathscr{F}(\mathcal{H})$ by using the injectivity of the Fourier transform to induce a norm on the Fresnel class: For $f \in \mathscr{F}(\mathcal{H})$ let $\mu_f$ denote the measure in $\mathscr{M}(\mathcal{H})$ with $\hat{\mu}_f = f$, then $\|f\|_{\mathscr{F}(\mathcal{H})} := \|\mu_f\|$ defines a norm on $\mathscr{F}(\mathcal{H})$. In this way, $\mathscr{F}(\mathcal{H})$ inherits the Banach property from $\mathscr{M}(\mathcal{H})$, and therefore the two spaces are isomorphic as Banach algebras with the isomorphism given by the Fourier transform.

Coming to the main result of this section, we return to finitely many dimensions, i.e. $\mathcal{H} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. We will show that any function $f$ in the Fresnel class $\mathscr{F}(\mathbb{R}^n)$ is Fresnel integrable and its Fresnel integral can be expressed in terms of an ordinary integral with respect to the corresponding measure $\mu_f \in \mathscr{M}(\mathbb{R}^n)$. It is a reformulation of [12, Prop. 2B] adapted to our setting.

**Theorem 1** *(Cameron-Martin type formula I). Any function $f \in \mathscr{F}(\mathbb{R}^n)$ is Fresnel integrable and its Fresnel integral is given by*

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(x)\, dx \;=\; (2\pi i \rho)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i\rho}{2}|x|^2}\, d\mu_f(x) \tag{2.11}$$

*where $\mu_f$ is the measure in $\mathscr{M}(\mathbb{R}^n)$ satisfying $\hat{\mu}_f = f$ and $i^{1/2}$ is taken to be the principle value of $\sqrt{i}$, that is $i^{1/2} := e^{i\pi/4}$.*

***Proof.*** Following the reasoning in [12], the strategy will be as follows: First, we want to show that (2.11) holds for Schwartz functions, where the oscillatory integral is replaced by an ordinary Lebesgue integral due to $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Second, we extend the result to products of functions from the Schwartz space and the Fresnel class, which will then allow us to show (2.11) for $f \in \mathscr{F}(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $\check{\varphi}$ denote its inverse Fourier transform, where constants are chosen as in (2.10), i.e. $\varphi(x) = \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} \check{\varphi}(y)\, dy$ for all $x \in \mathbb{R}^n$. The first thing we want to show, is

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} \varphi(x)\, dx \;=\; (2\pi i\rho)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i\rho}{2}|x|^2} \check{\varphi}(x)\, dx \tag{i}$$

Even though it's not needed in the proof, we remark that for Schwartz functions $\varphi$, we have $d\mu_\varphi(x) = \check{\varphi}(x)\, dx$, because by assumption, $\mu_\varphi$ has to fulfill $\hat{\mu}_\varphi = \varphi$ and the Fourier transform of the complex measure[2] $\check{\varphi}(x)\, dx$ coincides with the Fourier transform of the function $\check{\varphi}$, which is $\varphi$. The assertion then follows from the uniqueness of $\hat{\mu}_\varphi$. This shows that (i) is exactly (2.11) in the case of $f$ being a Schwartz function.

*Proof of (i):* In order to allow the application of Fubini's theorem in the next step below, we smuggle in $1 = \lim_{\varepsilon\to 0^+} e^{-\varepsilon|x|^2}$ on the l.h.s. of (i). We are allowed to apply Lebesgue's dominated convergence theorem, since $|e^{-\varepsilon|x|^2} e^{i|x|^2/2\rho}\varphi(x)| \leqslant |\varphi(x)|$ for all $x \in \mathbb{R}^n$ and $\varepsilon \geqslant 0$. Hence we obtain

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} \varphi(x)\, dx \;=\; \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} e^{-\varepsilon|x|^2} e^{\frac{i}{2\rho}|x|^2} \varphi(x)\, dx$$

By plugging in $\varphi(x) = \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} \check{\varphi}(y)\, dy$ we get

$$\int_{\mathbb{R}^n} e^{-\varepsilon|x|^2} e^{\frac{i}{2\rho}|x|^2} \varphi(x)\, dx \;=\; \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-(\varepsilon - \frac{i}{2\rho})|x|^2} e^{i\langle x,y\rangle}\, dx \right) \check{\varphi}(y) dy$$

where we used Fubini's theorem to interchange the order of integration, which is allowed due to $(x,y) \mapsto e^{-\varepsilon|x|^2}\check{\varphi}(y)$ being in $L^1(\mathbb{R}^{2n})$. Next, we use the standard result, that for any $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, the Fourier transform of the Gaussian exponential $e^{-\alpha|\cdot|^2}$ is given by

$$\int_{\mathbb{R}^n} e^{i\langle y,x\rangle} e^{-\alpha|x|^2} dx = \left(\frac{\pi}{\alpha}\right)^{n/2} e^{-\frac{1}{4\alpha}|y|^2} \tag{$*$}$$

[2]For any $f \in L^1(\mathbb{R}^n)$, $d\mu := f\, dx$ defines a complex measure $\mu \in \mathscr{M}(\mathbb{R}^n)$, since it has the polar decomposition $d\mu = h\, d|\mu|$ where $h$ is the phase factor in the polar decomposition of $f$, i.e. $f = h|f|$ and $d|\mu| = |f|\, dx$.

which can be computed in many different ways, e.g. by reducing to the one-dimensional case and then using contour integration [34]. Using $(*)$ with $\alpha = \varepsilon - \frac{i}{2\rho}$, the equation above gives

$$\int_{\mathbb{R}^n} e^{-\varepsilon|x|^2} e^{\frac{i}{2\rho}|x|^2} \varphi(x)\, dx \;=\; \left(\frac{2\pi\rho}{2\rho\varepsilon - i}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{\rho/2}{2\rho\varepsilon-i}|y|^2} \check{\varphi}(y)\, dy$$

which proves (i), since on the right-hand side the limit $\varepsilon \to 0^+$ can be taken inside of the integral by using the theorem of dominated convergence again. □(i)

Next, we will prove the following generalization of equation (i): For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathscr{F}(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} \varphi(x) f(x)\, dx \;=\; (2\pi i\rho)^{n/2} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i\rho}{2}|z|^2} \check{\varphi}(z-y)\, d\mu_f(y)dz \tag{ii}$$

For $f(x) \equiv 1$ this reduces to equation (i), since in this case the corresponding measure in $\mathscr{M}(\mathbb{R}^n)$ is given by $\mu_f = \delta_0$, the Dirac measure with mass at $0 \in \mathbb{R}^n$.

*Proof of (ii):* By plugging in the Fourier represention of $f$ and using Fubini's theorem to interchange the order of integration, we obtain

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} \varphi(x) f(x)\, dx = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2 + i\langle x,y\rangle} \varphi(x)\, dx \right) d\mu_f(y)$$

which after completing the square in the exponential and performing a change of variables is equal to

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|u|^2} \varphi(u-\rho y) du \right) e^{-\frac{i\rho}{2}|y|^2} d\mu_f(y)$$

For any fixed $\rho$ and $y$, the shifted function $\varphi_{\rho y} : u \mapsto \varphi(u - \rho y)$ is in $\mathcal{S}(\mathbb{R}^n)$, which allows us to apply (i) to the inner integral. Hence, we find

$$(2\pi i\rho)^{n/2} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-\frac{i\rho}{2}|u|^2} \check{\varphi}_{\rho y}(u)\, du \right) e^{-\frac{i\rho}{2}|y|^2} d\mu_f(y)$$

and due to $\check{\varphi}_{\rho y}(u) = e^{-i\langle u,\rho y\rangle} \check{\varphi}(u)$ an application of Fubini's theorem finally gives

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} \varphi(x) f(x)\, dx = (2\pi i\rho)^{n/2} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i\rho}{2}|u+y|^2} \check{\varphi}(u)\, d\mu_f(y)du$$

The change of variables $(y, u) \mapsto (y, u+y) =: (y, z)$ then proves the desired result. □(ii)

*Proof of the theorem:* Let $\varepsilon > 0$ and $\varphi \in \mathcal{S}^*$, i.e. $\varphi \in \mathbb{R}^n$ and $\varphi(0) = 1$ (compare Definition 1). As an application of equation (ii), we find

$$I_\varepsilon(\varphi, f) = \int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} f(x)\, \varphi(\varepsilon x)\, dx \;=\; \frac{(2\pi i\rho)^{n/2}}{\varepsilon^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i\rho}{2}|y|^2} \check{\varphi}\big(\tfrac{y-x}{\varepsilon}\big)\, d\mu_f(x)\, dy$$

where we also used the equality $\check{\varphi}_\varepsilon(x) = \check{\varphi}(\frac{x}{\varepsilon})/\varepsilon^n$ for all $x \in \mathbb{R}^n$. Hence, by performing another simple change of variables, we obtain

$$I_\varepsilon(\varphi, f) \;=\; (2\pi i\rho)^{n/2} \int e^{-\frac{i\rho}{2}|x+\varepsilon y|^2} \check{\varphi}(y)\, d\mu_f(x)\, dy$$

In order to prove the Fresnel integrability of $f$, we need to check whether $I_\varepsilon(\varphi, f)$ converges as $\varepsilon \to 0$ and whether its limit is independent of $\varphi \in \mathcal{S}^*$. Since the absolute value of the integrand is dominated by the function $(x, y) \mapsto |\check{\varphi}(y)|$, which is an element of $L^1(d\mu_f \times dy)$, an appliciation of Lebesgue's theorem of dominated convergence is justified. It follows

$$\lim_{\varepsilon\to 0} I_\varepsilon(\varphi, f) \;=\; (2\pi i\rho)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i\rho}{2}|x|^2} d\mu_f(x) \int_{\mathbb{R}^n} \check{\varphi}(y) dy \;<\; \infty$$

In particular, the limit exists and due to $\int \check{\varphi}(y)dy = \varphi(0) = 1$, its value is independent of $\varphi \in \mathcal{S}^*$. This proves that any $f \in \mathscr{F}(\mathbb{R}^n)$ is Fresnel integrable and its Fresnel integral is given by (2.11). □

The simplest example of a Fresnel class function $f \in \mathscr{F}(\mathbb{R}^n)$ is given by the constant function $f(x) = 1\ \forall x \in \mathbb{R}^n$, for which $\mu_f$ is just the Dirac measure centered at $x = 0$. Then the theorem shows $\int^o_{\mathbb{R}^n} e^{i|x|^2} dx = (\pi i)^{n/2}$.

## 2.3 Infinite-dimensional Fresnel integral

As was pointed out in the introduction, we are interested in a notion of Fresnel integration on *infinite-dimensional* Hilbert spaces, because in the application to quantum mechanics in section 2.4 we will show, how Fresnel integrals on special infinite-dimensional spaces of paths can be used to construct solutions to the Schrödinger equation for a point particle in Fresnel class potentials.

Actually, we are not going to use the same definition as S. Albeverio and R. Hoegh-Krohn used in their pioneering work [3], where they developed one of the first rigorous realizations of Feynman path integrals in terms of oscillatory integrals. For them, an equation similar to (2.11) served as *definition* of the Fresnel integral: For a function $f$ in the Fresnel class $\mathscr{F}(\mathcal{H})$ of a real separable Hilbert space $\mathcal{H}$, they set

$$\int_{\mathcal{H}} e^{\frac{i}{2}\|x\|^2} f(x)\, dx \;:=\; \int_{\mathcal{H}} e^{-\frac{i}{2}\|x\|^2} d\mu_f(x) \tag{2.12}$$

where $\mu_f \in \mathscr{M}(\mathcal{H})$ is the measure with $\hat{\mu}_f = f$. In other words, they used the right-hand side, which is just an ordinary integral with respect to a complex measure, to give meaning to the à priori ill-defined left-hand side.

Later, when the theory was further developed in several directions, another definition of infinite-dimensional oscillatory integrals was proposed. First introduced in [12] and developed in [1] in connection with an infinite-dimensional version of the method of stationary phase, it is now known as the *finite-dimensional approximation approach to infinite-dimensional oscillatory integrals* and—as its name lets suspect—it relies on finite-dimensional oscillatory integrals as defined in section 2.1 and a suitable limiting procedure.

In principle, this approach will define oscillatory integrals for a bigger class of functions, while on the Fresnel class it coincides with the original definition (Theorem 2). Even though we will stick with the Fresnel class for the biggest part of the thesis, we will still use the more general definition, since it relies on the well-established finite-dimensional theory and also favors the implementation of an infinite-dimensional stationary phase method.

Before we can get started with the infinite-dimensional theory, we first want to introduce a technical generalization of Definition 1, in order to cover general finite-dimensional inner product spaces. This work, as simple as it is, has not been done in the literature on the subject, even though it is needed for a rigorous treatment, in particular concerning the potential of confusion we address in Example 1.

Let $(W, \langle\cdot,\cdot\rangle)$ be a real inner product space with $n := \dim(W) < \infty$ and let $\gamma_E$ denote the isomorphism between $W$ and the Euclideon space $\mathbb{R}^n$, given by $\gamma_E : W \to \mathbb{R}^n$, $x \mapsto (x_k)_{k=1}^n$, where $x_n := \langle e_n, x\rangle$ denote the components of $x \in W$ with respect to a given orthonormal basis $E = (e_k)_{k=1}^n$ in $W$. Similar to the convention of neglecting the difference for a given function between being defined on a finite-dimensional linear space or on the corresponding isomorphic Eucliden space, we will allow for a slight abuse of notation, which is justified by the following lemma.

**Lemma 3** *(Independence from the choice of basis). Let $W$ be an $n$-dimensional real inner product space and $f$ a measurable function on $W$, such that there is a coordinate representation $f \circ \gamma_E^{-1}$, which is Fresnel integrable on $\mathbb{R}^n$. Then the Fresnel integral*

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(x))\, dx \tag{2.13}$$

*does not depend on the chosen orthonormal basis $E$, i.e. any coordinate representation of $f$ is Fresnel integrable and its Fresnel integral coincides with (2.13).*

***Proof.*** This is an application of Lemma 1. Let $E$ and $F$ be two orthonormal bases of $W$, then the coordinate representation of the transition map $\gamma_E \circ \gamma_F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal matrix $O$. By Lemma 1 the function $x \mapsto f(\gamma_E^{-1}(Ox))$ is Fresnel integrable on $\mathbb{R}^n$ and moreover

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(x))dx \overset{(2.5)}{=} \int_{\mathbb{R}^n}^{o} e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(Ox))dx$$

Since $\gamma_E^{-1}(Ox) = \gamma_E^{-1} \circ \gamma_E \circ \gamma_F^{-1}(x) = \gamma_F^{-1}(x)$ for all $x \in \mathbb{R}^n$, this proves the claim. □

**Definition 3** *(Fresnel integral over finite-dimensional inner product spaces).* We say that a function $f$ on an $n$-dimensional real inner product space $W$ is *Fresnel integrable*, whenever $f \circ \gamma_E^{-1}$ is Fresnel integrable on $\mathbb{R}^n$ for some orthonormal basis $E$ of $W$. Since, by the above Lemma, its Fresnel integral (2.13) does not depend on $E$, it is worthwhile to choose the more convenient notation

$$\int_W^o e^{\frac{i}{2\rho}\|x\|^2} f(x)\, dx \;:=\; \int_{\mathbb{R}^n}^o e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(x))\, dx \tag{2.14}$$

for the *Fresnel integral of* $f$.

As a remark, let us clarify the meaning of $dx$ in the expression on the left-hand side in (2.14). The most straight-forward way to carry the Lebesgue measure $\lambda$ in $\mathbb{R}^n$ over to an arbitrary inner product space $W$, is established by using the image measure of $\lambda$ under the map $\gamma_E^{-1} : \mathbb{R}^n \to W$ for a given orthonormal basis $E$ in $W$, which is given by $\lambda \circ \gamma_E$. Indeed, if we spell out the definition of the right-hand side of (2.14), we find it to be the limit of

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(x))\, \varphi(\varepsilon x)\, d\lambda(x) \;=\; \int_W e^{\frac{i}{2\rho}|\gamma_E(y)|^2} f(y)\, \varphi(\varepsilon\gamma_E(y))\, d(\lambda \circ \gamma_E)(y)$$

as $\varepsilon \to 0$, for any $\varphi \in \mathcal{S}^*$. Since $|\gamma_E(y)| = \|y\|$ for all $y \in W$, this identity shows that a Fresnel integral on $W$, as defined in (2.14), is the limit of a sequence of ordinary integrals on $W$ of the form

$$\int_W e^{\frac{i}{2\rho}\|x\|^2} f(x)\, \varphi_E(\varepsilon x)\, d(\lambda \circ \gamma_E)(x) \tag{2.15}$$

where $\varphi_E := \varphi \circ \gamma_E$. Hence, in comparison with the initial definition (Definition 1) of oscillatory integrals on $\mathbb{R}^n$ as limits of ordinary Lebesgue integrals of the same form as (2.15), the notation $dx$ in Fresnel integrals on $W$ can be seen to be connected to the image measure $\lambda \circ \gamma_E$, exactly in the same way, as the notation $dx$ in oscillatory integrals on $\mathbb{R}^n$ is connected to the Lebesgue measure $\lambda$. The importance of this difference can be seen in the following example.

**Example 1** *(Fresnel integral on $\mathbb{R}^n$ with a different inner product).* Let us consider the case, when $W$ is given by the Euclidean space $\mathbb{R}^n$, but equipped with the inner product given by the prescription $\langle x, y\rangle_A := \langle Ax, Ay\rangle_{\mathbb{R}^n}$, where $A$ is a real invertible $n \times n$ matrix. In view of Definition 3, we can write down the following Fresnel integral on $W = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$

$$\int_{(\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)}^o e^{\frac{i}{2\rho}|Ax|^2} f(x)\, dx \tag{2.16}$$

but as we shall see in a moment, despite its similar form, it is not exactly the same as $\int_{\mathbb{R}^n}^o e^{\frac{i}{2\rho}|Ax|^2} f(x)\, dx$ as introduced in Lemma 2, which is due to the different meaning of $dx$ in oscillatory integrals over different spaces.

If $\{e_i\}_{i=1}^n$ denotes the Euclidean orthonormal basis in $\mathbb{R}^n$, then $E := \{A^{-1}e_i\}_{i=1}^n$ forms an orthonormal basis in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$, and the map $\gamma_E$ is given by $A$. Therefore, by definition, expression (2.16) is equal to $\int_{\mathbb{R}^n}^o e^{\frac{i}{2\rho}|x|^2} f(A^{-1}x)\, dx$, which by Lemma 2 satisfies equation (2.6), i.e. it holds

$$\int_{(\mathbb{R}^n,\langle\cdot,\cdot\rangle_A)}^o e^{\frac{i}{2\rho}|Ax|^2} f(x)\, dx \;=\; |\det A\,| \int_{\mathbb{R}^n}^o e^{\frac{i}{2\rho}|Ax|^2} f(x)\, dx$$

In view of the above remark, this result is not a huge surprise, since the image of the Lebesgue measure $\lambda$ on $\mathbb{R}^n$ under the map $\gamma_E^{-1} = A^{-1}$ equals $\lambda \circ A = |\det A|\, \lambda$.

It is only a technicality to extend Theorem 1 to the case of oscillatory integrals on general finite-dimensional inner product spaces introduced in Definition 3.

**Proposition 2** *(Cameron-Martin type formula II). If $W$ is a real inner product space of dimension $n < \infty$, then any $f \in \mathscr{F}(W)$ is Fresnel integrable, and it holds*

$$\int_W^o e^{\frac{i}{2\rho}\|x\|^2} f(x)\, dx \;=\; (2\pi i\rho)^{n/2} \int_W e^{-\frac{i\rho}{2}\|x\|^2} d\mu_f(x) \tag{2.17}$$

*where $\|x\| = \langle x, x\rangle_W^{1/2}$ denotes the induced norm in $W$ and $\rho > 0$ is arbitrary.*

***Proof.*** By assumption, there exists $\mu_f \in \mathscr{M}(W)$, such that $f = \hat{\mu}_f$. By the unitarity of $\gamma_E$, it holds $\langle \gamma_E^{-1}(x), y\rangle_W = \langle x, \gamma_E(y)\rangle_{\mathbb{R}^n}$ for all $x \in \mathbb{R}^n$ and $y \in W$. If $d\mu_f = h\, d\nu$ denotes the polar decomposition of $\mu_f$, then

$$f \circ \gamma_E^{-1}(x) = \int_W e^{i\langle \gamma_E^{-1}(x), y\rangle_W} h(y)\, d\nu(y) = \int_{\mathbb{R}^n} e^{i\langle x, z\rangle} h(\gamma_E^{-1}(z))\, d(\gamma_E\nu)(z)$$

where $\gamma_E\nu := \nu \circ \gamma_E^{-1}$ is the image measure of $\nu$ under $\gamma_E$. Since $|(h \circ \gamma_E^{-1})| = 1$ and $\gamma_E\nu$ is finite, this shows that $f \circ \gamma_E^{-1} \in \mathscr{F}(\mathbb{R}^n)$ and Theorem 1 therefore implies that $f \circ \gamma_E^{-1}$ is Fresnel integrable and moreover

$$\int_{\mathbb{R}^n}^o e^{\frac{i}{2\rho}|x|^2} f(\gamma_E^{-1}(x))\, dx \;=\; (2\pi i\rho)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i\rho}{2}|x|^2}\, h(\gamma_E^{-1}(x))\, d(\gamma_E\nu)(x)$$

The left-hand side is the definition of the Fresnel integral of $f$ over $W$ and, since the integral on the right-hand side equals $\int_W e^{-\frac{i\rho}{2}\|x\|^2} h(x)\, d\nu(x)$, this proves the claim. □

From Theorem 1 and Proposition 2, we can see that the absolute value of Fresnel integrals—at least on the Fresnel class—grows exponentially with the dimension of the underlying space, e.g. for $f \equiv 1$, we have $|\int_{\mathbb{R}^n}^o e^{i|x|^2} dx| = \pi^{\,n/2}$. Since we want to define infinite-dimensional oscillatory integrals by approximating with finite-dimensional ones, we need to find a suitable normalization in order to get finite results. It turns out, that a good choice is, to give the oscillatory integral of $f \equiv 1$ the value 1.

**Definition 4** *(Normalized Fresnel integral).* Let $\mathcal{H}$ be a real separable Hilbert space and let $\mathscr{P}(\mathcal{H})$ denote the collection of all monotone increasing sequences of orthogonal projections $\mathcal{P}_n$ onto finite-dimensional subspaces of $\mathcal{H}$, which converge strongly to the identity, i.e. $\mathcal{P}_n x \to x$ as $n \to \infty$ for all $x \in \mathcal{H}$. Moreover, let

$$Z_{\rho,\mathcal{P}_n} := \int^{o}_{\mathcal{P}_n(\mathcal{H})} e^{\frac{i}{2\rho}\|x\|^2}\,dx \overset{(2.17)}{=} (2\pi i\rho)^{\dim \mathcal{P}_n(\mathcal{H})/2} \tag{2.18}$$

and for any function $f$ on $\mathcal{H}$, whose restriction[3] to $\mathcal{P}_n(\mathcal{H})$ is Fresnel integrable in the sense of Definition 3 for each $n \in \mathbb{N}$ and all $(\mathcal{P}_n)_{n\in\mathbb{N}} \in \mathscr{P}(\mathcal{H})$, set

$$\mathcal{F}^{\rho}_{\mathcal{P}_n}(f) := Z^{-1}_{\rho,\mathcal{P}_n} \int^{o}_{\mathcal{P}_n(\mathcal{H})} e^{\frac{i}{2\rho}\|x\|^2} f\big|_{\mathcal{P}_n(\mathcal{H})}(x)\,dx \tag{2.19}$$

If the sequence $(\mathcal{F}^{\rho}_{\mathcal{P}_n}(f))_{n\in\mathbb{N}}$ is convergent for all $(\mathcal{P}_n)_{n\in\mathbb{N}} \in \mathscr{P}(\mathcal{H})$ and $\rho > 0$, and if its limit is independent of the choice of $(\mathcal{P}_n)_{n\in\mathbb{N}}$, then we say that $f$ is *Fresnel integrable* and the limit

$$\mathcal{F}^{\rho}_{\mathcal{H}}(f) := \lim_{n\to\infty} \mathcal{F}^{\rho}_{\mathcal{P}_n}(f) \tag{2.20}$$

also denoted by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\rho}\|x\|^2} f(x)\,dx \tag{2.21}$$

will be referred to as the *(normalized) Fresnel integral* of $f$.

In the finite-dimensional case, $\mathcal{F}^{\rho}_{\mathcal{H}}(f)$ is just a normalized version of the Fresnel integral of $f$, introduced in Definition 3, because when $\dim \mathcal{H} < \infty$, for any sequence $(\mathcal{P}_n)_{n\in\mathbb{N}} \in \mathscr{P}(\mathcal{H})$, there is some $N \in \mathbb{N}$ such that $\mathcal{P}_n = \mathbb{1}$ for all $n \geqslant N$.

Apparently there is a slight notational confusion in the term *Fresnel integral.* In the following, if the underlying space is finite-dimensional, then we will use *oscillatory integral*, which refers to the unnormalized Fresnel integral of Definitions 1 and 3, whereas in the infinite-dimensional case, we will stick with *Fresnel integral*, since in this case it is clear that it refers to Definition 4.

The following theorem states the final Cameron-Martin type formula for the Fresnel integral on a general real separable Hilbert space $\mathcal{H}$. The original result was given in [12, 3C], but there it is only shown for the special case when $\mathcal{H}$ is given by the Cameron-Martin space $\mathcal{H}_t$ (see section 2.4). We provide a shorter proof, solely based on Definition 4, and Proposition 2.

[3] In the sense that the subspace $\mathcal{P}_n(\mathcal{H})$ is considered to be canonically embedded in $\mathcal{H}$.

**Theorem 2** *(Cameron-Martin type formula III – final version). Let $\mathcal{H}$ be a real separable Hilbert space of either finitely or infinitely many dimensions, then each $f \in \mathscr{F}(\mathcal{H})$ is Fresnel integrable on $\mathcal{H}$ and its normalized Fresnel integral is given by*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\rho}\|x\|^2} f(x)\, dx \;=\; \int_{\mathcal{H}} e^{-\frac{i\rho}{2}\|x\|^2} d\mu_f(x) \tag{2.22}$$

*where $\mu_f$ is the complex measure on $\mathcal{H}$, that satisfies $\hat{\mu}_f = f$.*

***Proof.*** In the finite-dimensional case, the claim is just a restatement of Proposition 2 without the constant $(2\pi i\rho)^{\dim \mathcal{H}/2}$, which is now part of the Fresnel integral.

If $\mathcal{H}$ is infinite-dimensional, we choose an arbitrary sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ from $\mathscr{P}(\mathcal{H})$ and let $\mathcal{H}_n$ denote the finite-dimensional subspace $\mathcal{P}_n(\mathcal{H})$. In order to see that functions of the form $f_n := f|_{\mathcal{H}_n}$ for $f \in \mathscr{F}(\mathcal{H})$ are in the corresponding Fresnel class $\mathscr{F}(\mathcal{H}_n)$, we may introduce the following notion (applying the reasoning in [7] on positive image measures to complex measures):

Claim *(complex image measure). Let $F : X \to Y$ be a measurable map between two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{A}')$. If $\mu$ is a complex measure on $(X, \mathcal{A})$, then the set function on $\mathcal{A}'$ given by $\mu \circ F^{-1}$ is a complex measure on $(Y, \mathcal{B})$, called the (complex) image measure of $\mu$ under $F$. Moreover, for any bounded measurable function $g$ on $Y$ it holds*

$$\int_Y g \; d(\mu \circ F^{-1}) \;=\; \int_X g \circ F \, d\mu \tag{2.23}$$

*Proof of the claim:* The countable additivity of $\mu \circ F^{-1}$ follows from the countable additivity of $\mu$, the measurability of $F$ and the properties of the preimage. More precisely, if $\{B_n\}_{n=1}^{\infty}$ is a countable partition of $B \in \mathcal{A}'$, then $\{F^{-1}(B_n)\}_{n=1}^{\infty}$ is a countable partition of $F^{-1}(B) \in \mathcal{A}$ and therefore $\mu \circ F^{-1}(B) = \mu(F^{-1}(B)) = \sum_n \mu(F^{-1}(B_n))$. Hence $\mu \circ F^{-1}$ is a complex measure on $(Y, \mathcal{A}')$.

By definition, equation (2.23) holds for each characteristic function $g = \mathbb{1}_E$ with $E \in \mathcal{A}'$ and by linearity it extends to simple functions. Since the positive measures $|\mu|$ and $|\mu \circ F^{-1}|$ are finite, any bounded measurable function $g$ is integrable with respect to $\mu$ and $\mu \circ F^{-1}$.

By linearity of (2.23), we only need to consider $g \geqslant 0$, because the general case of $g$ being complex-valued is obtained by decomposing $g$ into its real and imaginary positive and negative parts. Since the simple functions form a dense subset with respect to pointwise convergence in all non-negative measurable functions, there exists a monotone increasing sequence of simple functions $(g_n)_{n=1}^{\infty}$ converging pointwise to $g$. Hence by the theorem of dominated convergence, the right-hand side of (2.23) equals $\int_X (g \circ F) h \, d|\mu| = \lim_n \int_X (g_n \circ F) h \, d|\mu|$, where $d\mu = h\, d|\mu|$ denotes the polar representation of $\mu$. For the $g_n$ we already have established equation (2.23), so this shows $\int_X g \circ F \, d\mu = \lim_n \int_Y g_n \, d(\mu \circ F^{-1})$ and by applying the dominated convergence theorem again, we see that this limit equals the right-hand side of (2.23). □(claim)

*Proof of the theorem:* Using the claim on the projections $\mathcal{P}_n : \mathcal{H} \to \mathcal{H}_n$, we obtain for any $x \in \mathcal{H}_n$

$$f_n(x) = f(\mathcal{P}_n x) = \int_{\mathcal{H}} e^{i\langle \mathcal{P}_n x, y\rangle} d\mu_f(y) = \int_{\mathcal{H}} e^{i\langle x, \mathcal{P}_n y\rangle} d\mu_f(y) \overset{(2.23)}{=} \int_{\mathcal{H}_n} e^{i\langle x, z\rangle} d\mu_{f,n}(z)$$

where $\mu_{f,n} := \mu_f \circ \mathcal{P}_n^{-1}$ denotes the complex image measure of $\mu_f$ under $\mathcal{P}_n$. Hence $f_n \in \mathscr{F}(\mathcal{H}_n)$ and therefore by Proposition 2, $f_n$ is Fresnel integrable on $\mathcal{H}_n$ and

$$Z^{-1}_{\rho,\mathcal{P}_n} \int^{o}_{\mathcal{H}_n} e^{\frac{i}{2\rho}\|x\|^2} f_n(x)\, dx = \int_{\mathcal{H}_n} e^{-\frac{i\rho}{2}\|x\|^2}\, d\mu_{f,n}(x) = \int_{\mathcal{H}} e^{-\frac{i\rho}{2}\|\mathcal{P}_n x\|^2} d\mu_f(x)$$

Since the integrand in the very last integral is bounded by 1 and $|\mu_f|(\mathcal{H}) < \infty$, we can apply Lebesgue's dominated convergence theorem to take the limit $\lim_n \mathcal{F}^{\rho}_{\mathcal{P}_n}(f)$ inside of the integral. Any $(\mathcal{P}_n)_{n\in\mathbb{N}} \in \mathscr{P}(\mathcal{H})$ converges strongly to the identity and therefore we have $\|\mathcal{P}_n x\| \to \|x\|$ as $n \to \infty$, which on the one hand shows the independence of the limit from the choice of $(\mathcal{P}_n)_{n\in\mathbb{N}}$, i.e. $f$ is Fresnel integrable on $\mathcal{H}$, and moreover this proves equation (2.22). □

One property that can immediately be seen from the Cameron-Martin type formulas (2.11), (2.17) and (2.22), is the fact that $\mathcal{F}^{\rho}_{\mathcal{H}}$ is a bounded linear functional on the Fresnel class $\mathscr{F}(\mathcal{H})$, equipped with the total variation norm $\|f\| = \|\mu_f\|$ (section 2.2). Indeed

$$|\mathcal{F}^{\rho}_{\mathcal{H}}(f)| \overset{(2.22)}{\leqslant} |\mu_f|(\mathcal{H}) = \|f\| \tag{2.24}$$

Hence $\|\mathcal{F}^{\rho}_{\mathcal{H}}\|_{\mathscr{F}(\mathcal{H})\to\mathbb{C}} = 1$, since for $f = 1$, the value 1 is attained.

As mentioned earlier, Theorem 2 establishes the link between the initial definition of infinite-dimensional oscillatory Fresnel integrals by equation (2.12) as is given in [3], and the more recent one from [12], by means of approximations with finite-dimensional oscillatory integrals.

## 2.4 Application to Quantum Mechanics

In this section, we will show how normalized Fresnel integrals over suitable Hilbert spaces of paths can be used to construct solutions to the Schrödinger equation for a non-relativistic point particle in a Fresnel class potential.

More precisely, we consider the Hamiltonian $H = H_0 + V$, where $V$ denotes multiplication by a potential $V \in \mathscr{F}(\mathbb{R}^d)$ and $H_0$ generates the free evolution of a non-relativistic particle in $d$ dimensions.

In order to prevent distraction caused by unnecessary constants, throughout the thesis we will stick with natural units, i.e. $e = \hbar = 1$. Thus $H_0 = -\Delta/2$, where the Laplacian is taken on its natural domain of self-adjointness $\mathcal{D}(H_0) = H^2(\mathbb{R}^d)$, the second order Sobolev space on $\mathbb{R}^d$ [35, Theorem 7.8]. Since $V$ is bounded, it follows for example from Kato-Rellich, that $H$ is self-adjoint on $\mathcal{D}(H) := \mathcal{D}(H_0) = H^2(\mathbb{R}^d)$.

For the construction of a rigorous Feynman path integral formula based on the normalized Fresnel integral in Theorem 3, we need to introduce a proper Hilbert space of paths, the so called Cameron-Martin space $\mathcal{H}_t$.

**Definition 5** *(Cameron-Martin space).* For $t>0$, let $H^1(0,t;\mathbb{R}^d)$ denote the Sobolev space of square integrable $\mathbb{R}^d$-valued functions $x$ on $(0,t)$ with square integrable first order weak derivative $x'$. Due to the Sobolev embedding theorem in one dimension, any $x \in H^1(0,t;\mathbb{R}^d)$ can be represented by an absolutely continuous function on $[0,t]$. Thus

$$\mathcal{H}_t \;:=\; \{x \in H^1(0,t;\mathbb{R}^d) \,|\, x(t) = 0\} \tag{2.25}$$

defines a linear subspace of $H^1(0,t;\mathbb{R}^d)$, called *Cameron-Martin space.* We equip $\mathcal{H}_t$ with the inner product[4]

$$\langle x, y\rangle_t \;:=\; \langle x', y'\rangle_{L^2([0,t],\mathbb{R}^d)} \;=\; \int_{[0,t]} \langle x'(s), y'(s)\rangle_{\mathbb{R}^d}\, ds \tag{2.26}$$

The fact that $\mathcal{H}_t$ is indeed a separable Hilbert space is shown in the following result, which certainly can be found somewhere in the literature, but instead of searching for a proper reference, let us just give a proof by ourselves.

**Proposition 3**. *For each fixed $t > 0$, the Cameron-Martin space $\mathcal{H}_t$ equipped with the inner product (2.26) forms a real separable Hilbert space.*

***Proof.*** First of all, the induced norm $\|\cdot\|_{\mathcal{H}_t} = \langle\cdot,\cdot\rangle_t^{1/2}$ is weaker than the norm inherited from $H^1$, since $\|x\|^2_{\mathcal{H}_t} = \|x'\|^2_{L^2} \leqslant \|x\|^2_{L^2} + \|x'\|^2_{L^2} = \|x\|^2_{H^1}$. As a subspace of the separable normed space $H^1$, $\mathcal{H}_t$ is separable with respect to $\|\cdot\|_{H^1}$ by itself and therefore is also separable with respect to the weaker norm $\|\cdot\|_{\mathcal{H}_t}$. Moreover, $\mathcal{H}_t$ is closed in the $H^1$-norm, because it is the preimage of $\{0\} \subset \mathbb{R}^d$ under the linear map $x \mapsto x(t)$ from $H^1(0,t;\mathbb{R}^d)$ to $\mathbb{R}^d$, which happens to be bounded: To see this, let $x \in H^1$ and $t_m \in [0,t]$ be chosen such that $|x(t_m)| = \min_{s\in[0,t]} |x(s)|$, which exists since any $x \in H^1(0,t;\mathbb{R}^d)$ is bounded due to the Sobolev embedding theorem. Then absolute continuity gives

$$|x(t)| \;\leqslant\; |x(t_m)| + \int_{t_m}^t |x'(s)|\, ds \leqslant \frac{1}{t}\int_0^t |x(s)| ds + \int_{t_m}^t |x'(s)| ds$$

[4] The only property which is not inherited directly from the inner product in $L^2([0,t],\mathbb{R}^d)$ is the implication $\langle x, x\rangle = 0 \Rightarrow x = 0$: By the absolute continuity of $x$ and the initial condition $x(t) = 0$, we have for any $t' \in [0,t]$, $x(t') = x(t) - \int_{[t',t]} x'(s)ds = 0$, whenever $\|x'\|_{L^2} = 0$.

By setting $C_t := \max\{t^{1/2}, t^{-1/2}\}$, after applying Hölder's inequality, we obtain $|x(t)| \leqslant C_t\,(\|x'\|_{L^2} + \|x\|_{L^2})$. Due to the convexity of the square function, we have

$$2\left(\frac{\|x'\|_2}{2} + \frac{\|x\|_2}{2}\right)^2 \;\leqslant\; \|x'\|_2^2 + \|x\|_2^2 \;=\; \|x\|_{H^1}^2$$

Hence $|x(t)| \leqslant \sqrt{2}\,C_t\,\|x\|_{H^1}$ for any $x \in H^1([0,t],\mathbb{R}^d)$, so the linear map $x \mapsto x(t)$ is bounded and therefore continuous. Finally, as a closed subspace of $H^1$, $\mathcal{H}_t$ is complete with respect to $\|\cdot\|_{H^1}$ and thus also with respect to the weaker norm $\|\cdot\|_{\mathcal{H}_t}$. □

The following two examples show, how we can construct finite-dimensional projections on $\mathcal{H}_t$, which are needed for the definition of the normalized Fresnel integrals on $\mathcal{H}_t$. These results are indicated in [12] and [36] without providing a full proof.

**Example 2** *(Orthogonal projections on $\mathcal{H}_t$). For a family $\{\tau_j\}_{j=0}^N$ of points in $[0,t]$ with $\tau_j \leqslant \tau_{j+1}$ and $\tau_0 = 0$, $\tau_N = t$, let $\pi$ denote the corresponding partition of $[0,t]$ consisting of intervalls $[\tau_i,\tau_{j+1})$ and $\{t\}$. Moreover let $P_\pi : \mathcal{H}_t \to \mathcal{H}_t$ be the linear map, sending $x$ to the piecewise linear function on $[0,t]$, which has its nodes only in the $\tau_j$ and satisfies $(P_\pi x)(\tau_j) = x(\tau_j)$ for all $j = 0,\dots,N$, i.e. we have*

$$(P_\pi x)(s) = \sum_{j=0}^{N-1} \left( x(\tau_j) + \frac{x(\tau_{j+1}) - x(\tau_j)}{\tau_{j+1}-\tau_j}(s-\tau_j) \right) \mathbb{1}_{[\tau_j,\tau_{j+1})}(s) \qquad (2.27)$$

*for all $s \in [0,t]$. Then $P_\pi$ is a finite-dimensional orthogonal projection on $\mathcal{H}_t$.*

***Proof.*** First of all, it is clear that $\mathcal{H}_t^\pi := P_\pi(\mathcal{H}_t)$ is finite-dimensional, since it can be identified with the space of all $N$-tuples $(\xi_j)_{j=0}^{N-1}$ with $\xi_j \in \mathbb{R}^d$, by setting $\xi_j = x(\tau_j)$. Next, let us show that $P_\pi$ is bounded:

$$\|P_\pi x\|_t^2 \;=\; \sum_{j=1}^{N-1} \int_{\tau_j}^{\tau_{j+1}} \frac{|x(\tau_{j+1}) - x(\tau_j)|^2}{(\tau_{j+1}-\tau_j)^2} \;=\; \sum_{j=0}^{N-1} \frac{\left|\int_{\tau_j}^{\tau_{j+1}} x'(s)\,ds\right|^2}{\tau_{j+1}-\tau_j}$$

where we used that any $x \in \mathcal{H}_t$ is absolute continuous. By Hölder's inequality, we have $\int_{\tau_j}^{\tau_{j+1}} |x'(s)|dx \;\leqslant\; (\int_{\tau_j}^{\tau_{j+1}} |x'|^2)^{1/2}\,(\tau_{j+1}-\tau_j)^{1/2}$, and therefore

$$\|P_\pi x\|_t^2 \;\leqslant\; \sum_{j=0}^{N-1} \int_{\tau_j}^{\tau_{j+1}} |x'(s)|^2 ds \;=\; \|x\|_t^2$$

Hence $P_\pi$ is bounded on $\mathcal{H}_t$, with $\|P_\pi\| \leqslant 1$. The idempotence $P_\pi^2 = P_\pi$ follows directly from its definition, and for self-adjointness, we find for all $x, y \in \mathcal{H}_t$

$$\begin{aligned}\langle x, P_\pi y\rangle_t &= \sum_{j=0}^{N-1} \int_{\tau_j}^{\tau_{j+1}} x'(s) \cdot \left(\frac{y(\tau_{j+1}) - y(\tau_j)}{\tau_{j+1} - \tau_j}\right) ds \\ &= \sum_{j=0}^{N-1} \frac{1}{\tau_{j+1} - \tau_j} \int_{\tau_j}^{\tau_{j+1}} \left(\int_{\tau_j}^{\tau_{j+1}} x'(s) \cdot y'(u)\, du\right) ds \\ &= \sum_{j=0}^{N-1} \int_{\tau_j}^{\tau_{j+1}} \left(\frac{x(\tau_{j+1}) - x(\tau_j)}{\tau_{j+1} - \tau_j}\right) \cdot y(u)\, du \;=\; \langle P_\pi x, y\rangle_t\end{aligned}$$

where we used Fubini's theorem and absolute continuity again. This finishes the proof of $P_\pi$ being a finite-dimensional projection on $\mathcal{H}_t$. □

**Example 3** *(a suitable sequence of projections). For a monotone increasing function $N : \mathbb{N} \to \mathbb{N}$, let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of partitions of $[0, t]$, with each partition consisting of $|\pi_n| = N(n)$ intervals, such that*

$$\{\tau_j^n\}_{j=0}^{N(n)} \subset \{\tau_j^{n+1}\}_{j=0}^{N(n+1)} \;, \quad \Delta_n := \max_{j=1,\dots,N(n)} (\tau_j^n - \tau_{j-1}^n) \;\stackrel{n\to\infty}{\longrightarrow}\; 0$$

*where $\tau_j^n \in [0, t]$ denote the endpoints of the intervals in $\pi_n$. With these choices made, the monotone increasing sequence of projections, given by $(P_{\pi_n})_{n\in\mathbb{N}}$, as was defined in Example 2 converges strongly to the identity, i.e. it is in $\mathscr{P}(\mathcal{H}_t)$.*

***Proof.*** For $x \in \mathcal{H}_t \cap C^2([0, t])$ and any partition $\pi$ of $[0, t]$, we have

$$\|P_\pi x - x\|_t^2 \;=\; \int_0^t \left| \sum_{j=0}^{N-1} \left(\frac{x(\tau_{j+1}) - x(\tau_j)}{\tau_{j+1} - \tau_j} - x'(s)\right) \mathbb{1}_{[\tau_j, \tau_{j+1})}(s)\right|^2 ds$$

and by Taylor's theorem in Lagrange form, for each $j = 0, \dots, N - 1$

$$x'(s) = x'(\tau_j) + R_j(s) \;,\; x(\tau_{j+1}) = x(\tau_j) + x'(\tau_j)(\tau_{j+1} - \tau_j) + Q_j(\tau_{j+1})$$

where $R_j(s) = x''(u_j)(s-\tau_j)$ and $Q_j(\tau_{j+1}) = \frac{x''(v_j)}{2}(\tau_{j+1}-\tau_j)^2$ for suitable $u_j, v_j \in [0, t]$. Thus, the integrand of (2.28) is given by

$$\left| \sum_{j=0}^{N-1} \left(\frac{x''(v_j)}{2}(\tau_{j+1} - \tau_j) - x''(u_j)(s - \tau_j)\right) \mathbb{1}_{[\tau_j, \tau_{j+1})}(s)\right|^2$$

for all $s \in [0, t]$. By the triangle inequality, this is bounded from above by

$$\left(\sum_{j=0}^{N-1} C\, |\tau_{j+1} - \tau_j|\, \mathbb{1}_{[\tau_j, \tau_{j+1})}(s)\right)^2$$

where $C := 2\max_{s\in[0,t]}|x''(s)| \geqslant |x''(v_j)|/2 + |x''(u_j)|$ for all $j = 0,\dots,N-1$. Hence, setting $\Delta := \max_j |\tau_{j+1}-\tau_j|$, we've got

$$\begin{aligned}\|P_\pi x - x\|_t^2 \;&\leqslant\; C^2 \sum_{j=0}^{N-1}\sum_{i=0}^{N-1}(\tau_{j+1}-\tau_j)(\tau_{i+1}-\tau_i)\int_0^t \mathbb{1}_{[\tau_j,\tau_{j+1})\cap[\tau_i,\tau_{i+1})}(s)\,ds \\ &=\; C^2\sum_{j=0}^{N-1}(\tau_{j+1}-\tau_j)^3 \;\leqslant\; C^2\Delta^2\sum_{j=0}^{N-1}(\tau_{j+1}-\tau_j) \;=\; C^2\Delta^2 t\end{aligned}$$

For the partitions $\pi_n$ defined above, we therefore obtain $\|P_{\pi_n}x - x\|_t \leqslant C\sqrt{t}\,\Delta_n$, which converges to 0 as $n\to\infty$, for all $x\in\mathcal{H}_t\cap C^2([0,t])$.

Since $C^\infty([0,t])$ is dense in $H^1(0,t)$ with respect to the $H^1$-norm, and $\|x\|_t \leqslant \|x\|_{H^1(0,t)}$ for all $x\in\mathcal{H}_t$, also $\mathcal{H}_t\cap C^2([0,t])$ is dense in $\mathcal{H}_t$ with respect to $\|\cdot\|_t$. Therefore, we can carry over the result to the whole of $\mathcal{H}_t$, by performing a usual denseness argument: Let $\varepsilon>0$ and for a given $x\in\mathcal{H}_t$, choose $y\in\mathcal{H}_t\cap C^2([0,t])$ such that $\|x-y\|_t<\varepsilon/3$. Moreover, let $n_0\in\mathbb{N}$ be big enough, such that $\|P_{\pi_n}y-y\|_t<\varepsilon/3$ for all $n\geqslant n_0$ (by the above result). Then, by the standard $\varepsilon/3$-method, we obtain

$$\|P_{\pi_n}x-x\|_t \leqslant \|P_{\pi_n}(x-y)\|_t + \|P_{\pi_n}y-y\|_t + \|y-x\|_t < \frac{2}{3}\varepsilon+\frac{1}{3}\varepsilon=\varepsilon$$

for all $n\geqslant n_0$, where we used $\|P_{\pi_n}\|\leqslant 1$. Hence $P_{\pi_n}$ converges strongly to the identity in $\mathcal{H}_t$ and therefore belongs to $\mathscr{P}(\mathcal{H}_t)$. □

In the rest of this section, we will state and prove Theorem 3, which, as noted before, is the main result of Chapter 2, and properly interpreted, provides a rigorous Feynman path integral formula for solutions to the Schrödinger equation for a non-relativistic particle experiencing a Fresnel class potential, in terms of a normalized Fresnel integral on the Cameron-Martin space $\mathcal{H}_t$. Therefore, we will refer to (2.29) below as *Feynman-Fresnel path integral*, *FFPI* for short.

**Theorem 3** *(FFPI). If $\varphi\in\mathscr{F}(\mathbb{R}^d)\cap H^2(\mathbb{R}^d)$ and $V\in\mathscr{F}(\mathbb{R}^d)\cap C^2(\mathbb{R}^d)$ is real-valued with $\partial^\alpha V\in L^\infty(\mathbb{R}^d)$ for all $\alpha\in\mathbb{N}^d$ with $|\alpha|\leqslant 2$, then for any $t>0$ and all $\xi\in\mathbb{R}^d$, the function on $\mathcal{H}_t$, given by*

$$x \;\mapsto\; e^{-i\int_0^t V(x(s)+\xi)ds}\,\varphi(x(0)+\xi) \tag{2.28}$$

*belongs to $\mathscr{F}(\mathcal{H}_t)$. If $\mathcal{F}^\rho_{\mathcal{H}_t}$, in the case $\rho=1$, is denoted by $\mathcal{F}_t$, then by setting*

$$\psi_t(\xi) \;:=\; \mathcal{F}_t\Big(x\mapsto e^{-i\int_0^t V(x(s)+\xi)ds}\,\varphi(x(0)+\xi)\Big) \tag{2.29}$$

*we obtain $\psi_t\in H^2(\mathbb{R}^d)$ for all $t>0$. Moreover, the strong derivative of $t\mapsto\psi_t$ exists, and $\psi_t$ forms a solution to the Schrödinger equation*

$$\frac{d\psi_t}{dt} \;=\; -iH\psi_t \tag{2.30}$$

*with initial datum $\psi_0=\varphi$, and $H=H_0+V$, $\mathcal{D}(H)=H^2(\mathbb{R}^d)$.*

In order to see that (2.29) has indeed the form of a Feynman path integral, we recall that in our case, the classical action along a path $y \in H^1(0,t;\mathbb{R}^d)$ is given by $S(y) := \int_0^t \big(|y'(s)|^2/2 - V(y(s))\big)ds$. Therefore, by the definition of the inner product on $\mathcal{H}_t$, (2.29) informally reads

$$\psi_t(\xi) = \widetilde{\int_{\mathcal{H}_t}} e^{iS(x+\xi)}\,\varphi(x(0)+\xi)\,dx$$

where $x+\xi$ denotes the path $s \mapsto x(s)+\xi$ (with endpoint $\xi$). Thus $\psi_t(\xi)$ informally takes the form of a path integral as introduced by Feynman in [13].

The statement goes back to the pioneering work of Albeverio and Hoegh-Krohn [3], but their justifications are very short and miss a lot of details. Succeeding work on the topic (e.g. [12]) filled in some of the gaps, but for the taste of the author of this thesis, it still was a lot of work remaining unfinished.

We split up the whole proof into several intermediate results, and put them together at the end of the section (p. 45). References are given at the respective assertions, whenever they exist.

We start with the following Lemma, which provides two basic measure-theoretic results, which are taken from [22], and will serve as a preparation for a construction (which is due to [10]) of functions belonging to certain Fresnel classes in Lemma 5. It is then a simple application of that statement, to show that (2.28) belongs to $\mathscr{F}(\mathcal{H}_t)$ for all $t > 0$.

**Lemma 4**. *Let $(X,\mathcal{A})$ and $(Y,\mathcal{A}')$ be measurable spaces and let $\eta$ be either a non-negative $\sigma$-finite or a complex measure on $(X,\mathcal{A})$. For $\eta$-a.a. $x \in X$, let $\nu_x$ be a complex measure on $(Y,\mathcal{A}')$ such that for any $B \in \mathcal{A}'$, $x \mapsto \nu_x(B)$ is measurable. For $E \subset X \times Y$ and $x \in X$, let $E^{(x)}$ denote the $x$-section of $E$, i.e. $E^{(x)} := \{y \in Y \mid (x,y) \in E\}$.*

*(i) If $\theta$ is a bounded $\mathcal{A}\otimes\mathcal{A}'$-measurable function on $X \times Y$, then the function on $X$ given $\eta$-a.e. by $x \mapsto \int_Y \theta(x,y)\,d\nu_x(y)$ is $\mathcal{A}$-measurable.*

*(ii) If there exists $h \in L^1(X,\eta)$, such that $\|\nu_x\| \leqslant h(x)$ for $\eta$-a.e. $x \in X$, then the set function on $\mathcal{A}\otimes\mathcal{A}'$ given by $\mu(E) := \int_X \nu_x(E^{(x)})\,d\eta(x)$ is a complex measure on $X \times Y$ with $\|\mu\| \leqslant \|h\|_{L^1(X,\eta)}$. Moreover, we have the Fubini-type formula*

$$\int_X \Big(\int_Y \theta(x,y)d\nu_x(y)\Big)\,d\eta(x) \;=\; \int_{X\times Y} \theta(x,y)\,d\mu(x,y) \tag{2.31}$$

*where $\theta$ is any bounded $\mathcal{A}\otimes\mathcal{A}'$-measurable function on $X \times Y$.*

***Proof.*** The structure of the proof is as follows: In the first step, we consider the special case of $\theta(x,y) = \mathbb{1}_{E^{(x)}}(y)$ in (i). More precisely, by using the Monotone Class Theorem given in [17, §6 Theorem B], we will show that for any $E \in \mathcal{A}\otimes\mathcal{A}'$, the function $f_E$

given for $\eta$-a.a. $x \in X$ by $f_E(x) := \nu_x(E^{(x)})$ is $\mathcal{A}$-measurable. The usual procedure of extending the result first to simple functions, then to positive measurable ones and finally to the general case by considering positive and negative parts will finish the proof of (i) in step 2.

For the proof of (ii), the assumption $\|\nu_x\| \leqslant h(x)$ will be of central importance and will allow several applications of the dominated convergence theorem.

*Step 1 – measurability of $f_E$:* If $M := \{E \in \mathcal{A} \otimes \mathcal{A}' \mid f_E \text{ is measurable}\}$ is a monotone class (i.e. if it is closed under the limit of monotone sequences), which contains the algebra $\mathcal{R}$ of all measurable rectangles $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{A}'$, then by the Monotone Class Theorem, it follows that $M$ contains the $\sigma$-algebra generated by $\mathcal{R}$, which coincides with the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{A}'$.

If $A \times B \in \mathcal{R}$, then we have $(A \times B)^{(x)} = B$ whenever $x \in A$ and $(A \times B)^{(x)} = \emptyset$ for $x \neq A$ and therefore $\nu_x((A \times B)^{(x)}) = \nu_x(B)\mathbb{1}_A(x)$. Since by assumption, $x \mapsto \nu_x(B)$ is measurable, this shows $\mathcal{R} \subset M$.

To show that $M$ forms a monotone class, let $(E_n)_{n \in \mathbb{N}}$ be an increasing sequence in $M$ and let $E := \cup_{n=1}^{\infty} E_n$ denote its limit. Then for any $x \in X$, the sequence $(E_n^{(x)})_{n \in \mathbb{N}}$ is also monotone increasing and it holds $E^{(x)} = \cup_{n=1}^{\infty} E_n^{(x)}$. The usual continuity of non-negative measures with respect to increasing sequences also holds for complex measures, since it only relies on countable additivity. Thus $f_E(x) = \lim_n f_{E_n}(x)$ for $\eta$-a.a. $x \in X$, i.e. $f_E$ is the pointwise limit of measurable functions and therefore is measurable by itself. Hence $M$ is closed under the limit of monotone increasing sequences.

It is then completely analogous to show the closedness under limits of decreasing sequences, whereas in this case the continuity of $\nu_x$ follows from the continuity under limits of increasing sequences *and* its finiteness (but the finiteness of $\nu_x(E_n)$ for one $n \in \mathbb{N}$ would have been sufficient).

Altogether, we have $\mathcal{A} \otimes \mathcal{A}' = \sigma(\mathcal{R}) = M$ and therefore $x \mapsto \nu_x(E^{(x)})$ is measurable for all $E \in \mathcal{A} \otimes \mathcal{A}'$.

*Step 2 – proof of (i):* For indicators over measurable sets from $\mathcal{A} \otimes \mathcal{A}'$, (i) already was established in step 1, because for any $E \in \mathcal{A} \otimes \mathcal{A}'$, we have $\mathbb{1}_E(x, y) = \mathbb{1}_{E^{(x)}}(y)$. By linearity it directly extends to simple functions. For bounded $\mathcal{A} \otimes \mathcal{A}'$-measurable functions $\theta \geqslant 0$, let $(\theta_n)_n$ be a monotone increasing sequence of simple functions converging pointwise to $\theta$. If $d\nu_x = g\, d|\nu_x|$ is the polar decomposition of $\nu_x$, then $|\theta_n(x, y) g(y)| \leqslant \|\theta\|_\infty$ justifies the application of the theorem of dominated convergence in the integral

$$\int_Y \theta(x, y)\, d\nu_x(y) = \lim_{n \to \infty} \int_Y \theta_n(x, y)\, d\nu_x(y)$$

since $|\nu_x|$ is finite. Hence $x \mapsto \int_Y \theta(x, y)\, d\nu_x(y)$, as a pointwise limit of measurable functions, is $\mathcal{A}$-measurable. The complex case then follows by decomposition into real and imaginary positive and negative parts.

*Step 3 – $\mu$ is a complex measure with $\|\mu\| \leqslant \|h\|_1$:* For all $E \in \mathcal{A} \otimes \mathcal{A}'$, the measurable function on $X$ given $\eta$-a.e. by $x \mapsto \nu_x(E^{(x)})$ is in $L^1(X, \eta)$, since $|\nu_x(E^{(x)})| \leqslant \|\nu_x\| \leqslant$

$h(x)$ by assumption. Hence $|\mu(E)| < \infty$ for all $E \in \mathcal{A} \otimes \mathcal{A}'$. For countable additivity, let $\{E_n\}_{n=1}^{\infty}$ be a countable partition of $E$ in $\mathcal{A} \otimes \mathcal{A}'$, then $\{E_n^{(x)}\}_{n=1}^{\infty}$ is a countable partition of $E^{(x)}$ and $|\sum_{n=1}^{N} \nu_x(E_n^{(x)})| \leqslant \|\nu_x\| \leqslant h(x)$ for any $N \in \mathbb{N}$ and $\eta$-almost all $x \in X$. Thus, by the countable additivity of $\nu_x$ and the theorem of dominated convergence

$$\mu(E) = \int_X \Big( \sum_{n=1}^{\infty} \nu_x(E_n^{(x)}) \Big)\, d\eta(x) = \sum_{n=1}^{\infty} \int_X \nu_x(E_n^{(x)})\, d\eta(x) = \sum_{n=1}^{\infty} \mu(E_n)$$

Hence $\mu$ is a complex measure on $(X \times Y, \mathcal{A} \otimes \mathcal{A}')$. Moreover, for a countable collection $\{E_n\}_{n=1}^{\infty}$ of disjoint measurable sets in $\mathcal{A} \otimes \mathcal{A}'$ and any $N \in \mathbb{N}$, it holds

$$\sum_{n=1}^{N} |\mu(E_n)| \leqslant \sum_{n=1}^{N} \int_X |\nu_x(E_n^{(x)})|\, d|\eta|(x) = \int_X \sum_{n=1}^{N} |\nu_x(E_n^{(x)})|\, d|\eta|(x)$$

By the definition of the total variation measure $|\nu_x|$, we have $\sum_n |\nu_x(E_n^{(x)})| \leqslant \|\nu_x\|$ and therefore $\sum_{n=1}^{N} |\mu(E_n)| \leqslant \int_X \|\nu_x\|\, d|\eta|(x) \leqslant \|h\|_{L^1(X,\eta)}$. Again by the definition of the total variation measure, this shows $\|\mu\| = |\mu|(X) \leqslant \|h\|_{L^1(X,\eta)}$.

*Step 4 – Fubini-type formula:* First, we notice that due the assumption $\|\nu_x\| \leqslant h(x)$ for $\eta$-a.a. $x \in X$, we have $|\int_Y \theta(x,y)\, d\nu_x(y)| \leqslant \|\theta\|_\infty h(x)$ for any bounded $\mathcal{A} \otimes \mathcal{A}'$-measurable function $\theta$, i.e. the function $x \mapsto \int_Y \theta(x,y)\, d\nu_x(y)$ addressed by (i) now is even in $L^1(X,\eta)$.

For indicators $\theta = \mathbb{1}_E$, where $E \in \mathcal{A} \otimes \mathcal{A}'$, equation (2.31) reduces to the defintion of $\mu$, and by linearity it extends to simple functions. If $\theta$ is bounded and measurable, then let $(\theta_n)_{n=1}^{\infty}$ be a monotone increasing sequence of simple functions converging pointwise to $\theta$ and let $d\mu = g\, d|\mu|$ be the polar representation of $\mu$. Due to $|\theta_n(x,y)g(x,y)| \leqslant \|\theta\|_\infty$ and the finiteness of $|\mu|$, the theorem of dominated convergence gives

$$\int_{X \times Y} \theta\, d\mu = \lim_{n\to\infty} \int_{X\times Y} \theta_n\, d\mu = \lim_{n\to\infty} \int_X \Big( \int_Y \theta_n(x,y)\, d\nu_x(y) \Big) d\eta(x)$$

As above, for $\eta$-a.a. $x \in X$, we have $|\int_Y \theta_n(x,y)\, d\nu_x(y)| \leqslant \|\theta\|_\infty h(x)$. Again, the assumption $h \in L^1(X,\eta)$, $|\theta_n| \leqslant \|\theta_n\|_\infty$, and the finiteness of $|\nu_x|$ allow us to apply the theorem of dominated convergence twice in order to take the limit first inside of the outer and then inside of the inner integral:

$$\int_{X \times Y} \theta\, d\mu = \int_X \Big( \lim_{n\to\infty} \int_Y \theta_n(x,y)\, d\nu_x(y) \Big)\, d\eta(x) = \int_Y \Big( \int_X \theta(x,y)\, d\nu_x(y) \Big)\, d\eta(x)$$

By linearity of (2.31), decomposing into real and imaginary positive and negative parts extends the result to the case of general bounded measurable functions. □

**Lemma 5** *(a construction of Fresnel class functions). Let $\mathcal{H}$ be a real separable Hilbert space, $(X, \mathcal{A}, \eta)$ a measure space, where $\eta$ is either a $\sigma$-finite non-negative or a complex* *measure let $d \in \mathbb{N}$, and for $i = 1, \ldots, d$ let $\phi_i : X \to \mathcal{H}$ be measurable functions, where $\mathcal{H}$ is equipped with the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H})$. If $\{\nu_x\}_{x \in X}$ is a family of complex measures on $\mathbb{R}^d$, such that $x \mapsto \nu_x(B)$ is measurable for every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$, then the function $x \mapsto \|\nu_x\|$ is measurable. Further, if $\hat{\nu}_x$ denotes the Fourier transform of $\nu_x$ and if $x \mapsto \|\hat{\nu}_x\|_{\mathscr{F}(\mathbb{R}^d)}$ is in $L^1(X, \eta)$, then also $x \mapsto \hat{\nu}_x((\langle y, \phi_j(x)\rangle)_{j=1,\ldots,d})$ is in $L^1(X, \eta)$ for each $y \in \mathcal{H}$ and moreover, the function on $\mathcal{H}$, defined by*

$$f(y) := \int_X \hat{\nu}_x\big(\langle y, \phi_1(x)\rangle, \ldots, \langle y, \phi_d(x)\rangle\big)\, d\eta(x) \tag{2.32}$$

*belongs to $\mathscr{F}(\mathcal{H})$ and satisfies $\|f\|_{\mathscr{F}(\mathcal{H})} \leqslant \int_X \|\nu_x\|\, d|\eta|(x)$.*

***Proof.*** As mentioned above, this result is due to [10] and most of its proof relies on Lemma 4. In the following, let $C_0(\mathbb{R}^d)$ denote the space of continuous functions $f$ on $\mathbb{R}^d$ vanishing at infinity, i.e. for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that $|f(x)| < \varepsilon$ for all $x \notin K$.

*Step 1 – measurability of $x \mapsto \|\nu_x\|$:* By the Riesz-Markov Theorem [31, 6.19], the dual space $C_0(\mathbb{R}^d)^*$, consisting of all bounded linear functionals on the Banach space $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$, is isomorphically equivalent to the space of regular complex Borel measures on $\mathbb{R}^d$, which coincides with the space $\mathcal{M}(\mathbb{R}^d)$ of all complex Borel measures, as it is the case for any metric space [27, p. 27]. More precisely, the isomorphism $\varphi$ between $\mathscr{M}(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)^*$ is given by $\varphi(\mu) : C_0(\mathbb{R}^d) \to \mathbb{C}$, $f \mapsto \int_{\mathbb{R}^d} f\, d\mu$ for all $\mu \in \mathscr{M}(\mathbb{R}^d)$ and it satisfies $\|\varphi(\mu)\|_{C_0(\mathbb{R}^d)^*} = \|\mu\|_{\mathscr{M}(\mathbb{R}^d)}$. Thus, if $B_1$ denotes the unit ball in $C_0(\mathbb{R}^d)$, then

$$\|\nu_x\| = \sup_{f \in B_1} |\varphi(\nu_x) f|$$

for any $x \in X$. By (i) of Lemma 4, the functions $x \mapsto \int_{\mathbb{R}^d} f\, d\nu_x = \varphi(\nu_x) f$ are measurable for all $f \in C_0(\mathbb{R}^d)$. It is a corollary of the Stone-Weierstrass Theorem [11, 2.4.11], that $C_0(\mathbb{R}^d)$ is separable and therefore the unit ball $B_1$ contains a countable set $D$, which is dense in the sup-norm. Thus $x \mapsto \|\nu_x\| = \sup_{f \in D} |\varphi(\nu_x) f|$, as the supremum of measurable functions over a countable set, is measurable by itself.

*Step 2 – measurability of $x \mapsto \hat{\nu}_x((\langle y, \phi_j(x)\rangle)_{j=1,\ldots,d})$:* First, we apply (i) of Lemma 4 again, to show that $(x, k) \mapsto \hat{\nu}_x(k) = \int_{\mathbb{R}^d} e^{iku} d\nu_x(u)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. The role of $(Y, \mathcal{A}')$ in the Lemma, here is played by $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and if we define $\nu_{(x,k)} := \nu_x$ for all $k \in \mathbb{R}^d$, then for $(X, \mathcal{A})$ replaced by $(X \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d))$ Lemma 4 applies to $\theta((x, k), u) := e^{iku}$ and so, it proves the measurability of $\hat{\nu}_x(k)$ as a function of $(x, k)$.

Now, since by assumption the functions $\phi_i$ are measurable, for each fixed $y \in \mathcal{H}$ the map $x \mapsto \hat{\nu}_x((\langle y, \phi_j(x)\rangle)_{j=1,\ldots,d})$ can be represented as a composition of measurable functions and therefore is measurable.

*Step 3 – construction of $\mu$ with $\hat{\mu} = f$:* Due to Lemma 4 and the additional assumption on the function $x \mapsto \|\nu_x\| = \|\hat{\nu}_x\|_{\mathscr{F}(\mathbb{R}^d)}$ of being in $L^1(X, \eta)$, the set function $\rho$ on $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ given by $\rho(E) = \int_X \nu_x(E^{(x)})\, d\eta(x)$ is a complex measure, and it holds $\|\rho\| \leqslant \int_X \|\nu_x\|\, d|\eta|(x)$. Now, we are ready to construct the complex measure $\mu$ on $\mathcal{H}$

with the desired property $\hat{\mu} = f$. If we define a measurable map $F$ between $X \times \mathbb{R}^d$ and $\mathcal{H}$ by $F(x,v) := \sum_i v_i \phi_i(x)$, then $\mu$ is given by $\rho \circ F^{-1}$, the complex image measure[5] of $\rho$ under $F$. Indeed

$$\hat{\mu}(y) = \int_{\mathcal{H}} e^{i\langle y,z\rangle}\, d(\rho \circ F^{-1})(z) \overset{(2.23)}{=} \int_{X\times\mathbb{R}^d} e^{i\langle y,F(x,v)\rangle} d\rho(x,v)$$

for any fixed $y \in \mathcal{H}$ and applying (2.31) to $\theta(x,v) := e^{i\langle y,F(x,v)\rangle} = e^{i\sum_i v_i\langle y,\phi_i(x)\rangle}$ gives

$$\begin{aligned}\hat{\mu}(y) &= \int_X \Big(\int_{\mathbb{R}^d} e^{i\sum_i v_i\langle y,\phi_i(x)\rangle}\, d\nu_x(v)\Big) d\eta(x)\\ &= \int_X \hat{\nu}_x\big(\langle y,\phi_1(x)\rangle,\ldots,\langle y,\phi_d(x)\rangle\big) d\eta(x)\end{aligned}$$

which coincides with the definition of $f$. By the definition of $\|\cdot\|_{\mathscr{F}(\mathcal{H})}$, this also completes the proof of the estimate $\|f\|_{\mathscr{F}(\mathcal{H})} = \|\mu\| \leqslant \|\rho\| \leqslant \int_X \|\nu_x\|\, d|\eta|(x)$. □

As a simple application of Lemma 5, we obtain the following, which is also taken from [10].

**Proposition 4**. *For $t > 0$ let $\mathcal{H}_t$ denote the Cameron-Martin space (2.25). Then for any $\psi \in \mathscr{F}(\mathbb{R}^d)$ and $u \in [0,t]$, the functions $f$ and $g$ on $\mathcal{H}_t$, given by*

$$f(x) := \psi(x(u)) \ , \quad g(x) := \int_u^t \psi(x(s))\, ds \tag{2.33}$$

*belong to $\mathscr{F}(\mathcal{H}_t)$.*

***Proof.*** First, for $\tau \in [0,t]$, we introduce a family of paths $\{\gamma_\tau^i\}_{i=1}^d \subset \mathcal{H}_t$ with the property $\langle x,\gamma_\tau^i\rangle_t = x_i(\tau)$ for all $x \in \mathcal{H}_t$ and $i = 1,\ldots,d$. More precisely, the $j$-th component of $\gamma_\tau^i$ is given by $(\gamma_\tau^i(s))_j := (t - s \vee \tau)\delta_{ij}$ for all $s \in [0,t]$, where $s \vee \tau := \max\{s,t\}$. Hence the $j$th component of $\gamma_\tau^i$ has the weak derivative $-\delta_{ij}\mathbb{1}_{[\tau,t]}$ and we have $\gamma_\tau^i(t) = 0$, $\gamma_\tau^i \in H^1(0,t;\mathbb{R}^d)$, i.e. $\gamma_\tau^i \in \mathcal{H}_t$ for all $i = 1,\ldots,d$ and $\tau \in [0,t]$, and moreover

$$\langle x,\gamma_\tau^i\rangle_t = -\int_0^t \mathbb{1}_{[\tau,t]}(s)\, x_i'(s)\, ds = -x_i(t) + x_i(\tau) = x_i(\tau)$$

We apply Lemma 5 with $\mathcal{H} = \mathcal{H}_t$, $(X,\mathcal{A},\eta) = ([0,t],\mathcal{B}([0,t]),\mathcal{P})$, where $\mathcal{P}$ is an arbitrary probability measure on $[0,t]$, $\hat{\nu}_s = \psi$ and $\phi_i(s) = \gamma_u^i$ for all $s \in [0,t]$ and $i = 1,\ldots,d$. With these choices, the requirements of Lemma 5 are trivially fulfilled and the function

[5]Complex image measures were introduced in the proof of Theorem 2.

(2.32), evaluated at $x \in \mathcal{H}_t$, is equal to

$$\int_{[0,t]} \psi(\langle x, \gamma_u^1 \rangle, \dots, \langle x, \gamma_u^d \rangle)\, d\mathcal{P}(s) = \psi(x_1(u), \dots, x_d(u))\, \mathcal{P}([0,t]) = \psi(x(u)) = f(x)$$

So $f$ belongs to $\mathscr{F}(\mathcal{H}_t)$. In order to get the result for $g$, we make the same choices as above, except for $\eta$, which is taken to be the Lebesgue measure on $[u,t]$ extended by 0 to $[0,t]$ and the $\phi_i$, which are now defined as $\phi_i(\tau) = \gamma_\tau^i$. In this case, the only non-trivial fact we need to check in order to fulfill the requirements of Lemma 5, is Borel measurability of the $\phi_i$, which is easiest seen by proving their continuity. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence in $[0,t]$ that converges to $\tau \in [0,t]$, then for each $i = 1, \dots, d$

$$\|\gamma_{\tau_n}^i - \gamma_\tau^i\|_t^2 = \int |\mathbb{1}_{[\tau_n,t]}(s) - \mathbb{1}_{[\tau,t]}(s)|\, ds = \int_{\min\{\tau_n,\tau\}}^{\max\{\tau_n,\tau\}} ds = |\tau_n - \tau|$$

Hence, as $\tau_n$ approaches $\tau$, $\gamma_{\tau_n}^i$ converges in $(\mathcal{H}_t, \|\cdot\|_t)$ to $\gamma_\tau^i$, i.e. the $\phi_i$ are continuous and therefore Borel measurable. In this case, (2.32) equals

$$\int_{[u,t]} \psi(\langle x, \gamma_s^1 \rangle, \dots, \langle x, \gamma_s^d \rangle)\, ds = \int_u^t \psi(x(s))\, ds = g(x)$$

and therefore also $g$ is in $\mathscr{F}(\mathcal{H}_t)$. □

The next result is not found in the literature, but crutial for the proof of Theorem 3. It does not follow from Proposition 6, where it is shown that on the Schwartz space, $U_0(t)|_{\mathcal{S}(\mathbb{R}^d)} = e^{-itH_0}$, since for an application of the bounded extension principle, we already have to know that $U_0(t)$ is bounded on its domain of definition.

**Proposition 5** *($L^2$-Boundedness of a special form of Fresnel integrals). If the normalized Fresnel integral $\mathcal{F}_{\mathcal{H}_t}^\rho$ in the case $\rho = 1$ is denoted by $\mathcal{F}_t$, then for each $t > 0$ the linear map $U_0(t)$ on $(\mathscr{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \|\cdot\|_2)$, defined by*

$$(U_0(t)\varphi)(\xi) := \mathcal{F}_t\Big(x \mapsto \varphi(x(0)+\xi)\Big) \tag{2.34}$$

*maps into $L^2(\mathbb{R}^d)$, is bounded, and does not exceed 1 in operator norm.*

***Proof.*** First of all, as a consequence of Proposition 4, the function on $\mathcal{H}_t$ given by $x \mapsto \varphi(x(0) + \xi)$ belongs to $\mathscr{F}(\mathcal{H}_t)$, whenever $\varphi \in \mathscr{F}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$, since $\mathscr{F}(\mathbb{R}^d)$ is closed under shifts in the coordinates. Hence, the argument of the Fresnel integral (2.34) is a product of Fresnel class functions and therefore also an element of $\mathscr{F}(\mathcal{H}_t)$ by itself.

Now, let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,t]$ with $|\pi_n| = n$ and let $P_{\pi_n}$ denote the corresponding finite-dimensional projections from $\mathcal{H}_t$ to the space $\mathcal{H}_t^{\pi_n}$ of piecewise linear functions, as they were introduced in Examples 2 and 3. Due to the results shown there, the finite-dimensional Fresnel integrals on $\mathcal{H}_t^{\pi_n}$ given by

$$u_t^n(\xi) \;:=\; (2\pi i)^{-nd/2} \int_{\mathcal{H}_t^{\pi_n}}^{o} e^{\frac{i}{2}\|x\|_t^2}\, \varphi(x(0)+\xi)\, dx$$

form a pointwise approximation of $U_0(t)\varphi$, as $n \to \infty$. In order to compute Fresnel integrals on $\mathcal{H}_t^{\pi_n}$, we use the orthonormal basis $E$ in $\mathcal{H}_t^{\pi_n}$ given by

$$E \;:=\; \Big\{\alpha_k^j \in \mathcal{H}_t^{\pi_n} \;\Big|\; \alpha_k^j = \frac{\gamma_{t_{k+1}}^j - \gamma_{t_k}^j}{(\Delta_k t)^{1/2}},\, 1 \leqslant j \leqslant d,\, 0 \leqslant k \leqslant n-1\Big\}$$

where $\Delta_k t := t_{k+1} - t_k$ and $t_k$ denote the endpoints of the intervalls in $\pi_n$. Orthonormality follows directly from the defining property of the $\gamma_\tau^j$, i.e. $\langle \gamma_\tau^j, x\rangle_t = x_j(\tau)$, for any $x \in \mathcal{H}_t$, $\tau \in [0,t]$ and $j = 1,\dots,d$ (see the proof of Proposition 4). Moreover, by definition, the $i$th component of $\alpha_k^j(s)$ is given by $\delta_{ij}(\tau_k \vee s - \tau_{k+1} \vee s)/(\Delta_k\tau)^{1/2}$. For $s < \tau_k$ this equals $-\delta_{ij}(\Delta_k\tau)^{1/2}$, while for $s \in [\tau_k, \tau_{k+1}]$, $(\alpha_k^j(s))_i = \delta_{ij}(s - \tau_{k+1})/(\Delta_k\tau)^{1/2}$, and for $s < \tau_{k+1}$, $\alpha_k^j(s) = 0$. Since $\mathcal{H}_t^{\pi_n}$ is the image of $\mathcal{H}_t$ under the projection $P_{\pi_n}$, which is given by (2.27), the $j$th component of any $x \in \mathcal{H}_t^{\pi_n}$ can be written as

$$x_j(s) \;=\; (P_{\pi_n}x)_j(s) \;=\; \sum_{k=0}^{n-1} \bigg( x_j(\tau_k) + \frac{s-\tau_k}{\tau_{k+1}-\tau_k}\big(x_j(\tau_{k+1}) - x_j(\tau_k)\big)\bigg)\, \mathbb{1}_{[\tau_k,\tau_{k+1})}(s) \tag{$*$}$$

In order to show, that this takes the form of a linear combination of the $\alpha_k^j \in E$, we start from the expression $\sum_{j=1}^d \sum_{k=0}^{n-1} \langle \alpha_k^j, x\rangle_t\, \alpha_k^j(s)$. By plugging in the explicit values of $\alpha_k^j(s)$, we obtain

$$\sum_{j=0}^{d}\sum_{k=0}^{n-1} \langle \alpha_k^j, x\rangle_t\, \alpha_k^j(s) \;=\; \sum_{k=0}^{n-1}\bigg(\frac{\Delta_k x}{\Delta_k \tau}\,(s-\tau_{k+1})\,\mathbb{1}_{[\tau_k,\tau_{k+1})}(s) - \Delta_k x\; \mathbb{1}_{[0,\tau_k)}(s)\bigg)$$

where $\Delta_k x := x(\tau_{k+1}) - x(\tau_k)$. A small calculation shows

$$\sum_{k=0}^{n-1} \Delta_k x\; \mathbb{1}_{[0,\tau_k)} \;=\; \sum_{l=0}^{n-1}\bigg(\sum_{k=l}^{n-1}\Delta_k x\bigg)\mathbb{1}_{[\tau_l,\tau_{l+1})} \;=\; -\sum_{k=0}^{n-1}\Big(\Delta_k x + x(\tau_k)\Big)\,\mathbb{1}_{[\tau_k,\tau_{k+1})}$$

and together with $\frac{\Delta_k x}{\Delta_k \tau}(s-\tau_{k+1}) + \Delta_k x = \frac{\Delta_k x}{\Delta_k \tau}(s-\tau_k)$, we obtain the identity

$$\sum_{j=0}^{d}\sum_{k=0}^{n-1} \langle \alpha_k^j, x\rangle_t\, \alpha_k^j(s) \;=\; \sum_{k=0}^{n-1}\bigg(x(\tau_k) + \frac{\Delta_k x}{\Delta_k\tau}(s-\tau_k)\bigg)\,\mathbb{1}_{[\tau_k,\tau_{k+1})}(s) \;\stackrel{(*)}{=}\; x(s)$$

which finishes the proof of $E$ being an orthonormal basis of $\mathcal{H}_t^{\pi_n}$. Hence, by definition, $u_t^n(\xi)$ is given by the oscillatory Fresnel integral on $\mathbb{R}^{nd}$

$$u_t^n(\xi) \;=\; (2\pi i)^{-nd/2} \int_{\mathbb{R}^{nd}}^{o} e^{\frac{i}{2}|y|^2}\, \varphi((\gamma_E^{-1}y)(0)+\xi)\, dy$$

where $\gamma_E^{-1}$ is the inverse of $\gamma_E : \mathcal{H}_t^{\pi_n} \to \mathbb{R}^{nd}$, the coordinate map corresponding to $E$, i.e. $\gamma_E(x) = (\langle \alpha_k^j, x\rangle_t)_{j,k}$. Let us switch from this set of coordinates, which are explicitly given by $y_k^j = (\Delta_k x_j)/(\Delta_k \tau)^{1/2}$, to the set of points $\eta_k = x(\tau_k) \in \mathbb{R}^d$, $k = 0, \dots, n-1$. For this, let $A$ be the $nd \times nd$ block matrix consisting of $n^2$ square matrices $A_{kl}$ of dimension $d \times d$, where the only non-zero blocks are given by $A_{kk} = -(\Delta_k \tau)^{-1/2} I_d$ and $A_{kk+1} = (\Delta_k \tau)^{-1/2} I_d$ for all $k = 0, \dots, n-1$ ($I_d$ denotes the identity matrix on $\mathbb{R}^d$). Then it holds

$$A(\eta_0, \dots, \eta_{n-1}) = \Big( \frac{\eta_1 - \eta_0}{(\Delta_0 \tau)^{1/2}}, \dots, \frac{\eta_n - \eta_{n-1}}{(\Delta_{n-1}\tau)^{1/2}} \Big) \ \in \ \big(\mathbb{R}^d\big)^n$$

where $\eta_n := 0$. By Lemma 2, changing variables from $y$ to $\eta$ gives

$$u_t^n(\xi) \ = \ (2\pi i)^{-nd/2} \prod_{k=0}^{n-1} (\Delta_k \tau)^{-d/2} \int_{(\mathbb{R}^d)^n}^{o} e^{\frac{i}{2}|A\eta|^2} \varphi(\eta_0 + \xi)\, d\eta$$

If $\phi \in \mathcal{S}^*(\mathbb{R}^d)$ $(= \mathcal{S} \cap \{\phi \,|\, \phi(0) = 1\})$, then the product $\phi^{\otimes n}(\eta) := \phi(\eta_0)\phi(\eta_1) \cdots \phi(\eta_{n-1})$ belongs to $\mathcal{S}^*((\mathbb{R}^d)^n)$ and therefore qualifies to be used in the definition (Lemma 2) of the oscillatory integral $\int_{(\mathbb{R}^d)^n}^{o} e^{\frac{i}{2}|A\eta|^2} \varphi(\eta_0 + \xi)\, d\eta$. Hence, for each $\xi \in \mathbb{R}^d$, $u_t^n(\xi)$ is the limit of

$$u_t^{n,\varepsilon}(\xi) \ := \ \lambda_n^{-1} \int_{(\mathbb{R}^d)^n} \left( \prod_{k=0}^{n-1} e^{\frac{i}{2}|\eta_{k+1} - \eta_k|^2/\Delta_k \tau} \right) \phi(\varepsilon \eta_{n-1}) \cdots \phi(\varepsilon \eta_0)\, \varphi(\eta_0 + \xi)\, d\eta$$

as $\varepsilon \to 0$, where we have set $\lambda_n := (2\pi i)^{nd/2} \prod_{k=0}^{n-1} (\Delta_k \tau)^{d/2}$. Since the absolute value of the integrand in this expression has the function $\|\varphi\|_\infty |\phi_\varepsilon|^{\otimes n} \in L^1((\mathbb{R}^d)^n)$ as upper bound ($\phi_\varepsilon(x) := \phi(\varepsilon x)$), we are allowed to apply Fubini's theorem in order to obtain

$$\begin{aligned} u_t^{n,\varepsilon}(\xi) \ = \ \lambda_n^{-1} & \int_{\mathbb{R}^d} e^{\frac{i}{2}|\xi - x_{n-1}|^2/\Delta_{n-1}\tau} \psi_\varepsilon^\xi(x_{n-1}) \Big( \int_{\mathbb{R}^d} e^{\frac{i}{2}|x_{n-1} - x_{n-2}|^2/\Delta_{n-2}\tau} \psi_\varepsilon^\xi(x_{n-2}) \\ & \cdots \Big( \int_{\mathbb{R}^d} e^{\frac{i}{2}|x_1 - x_0|^2/\Delta_0 \tau} \psi_\varepsilon^\xi(x_0) \varphi(x_0) dx_0 \Big) \cdots dx_{n-2} \Big) dx_{n-1} \end{aligned}$$

after setting $\psi_\varepsilon^\xi(x_k) := \phi(\varepsilon(x_k - \xi))$ which satisfies $\|\psi_\varepsilon^\xi\|_\infty = \|\phi\|_\infty$ for all $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, and applying the change of variables $x_k = \eta_k + \xi$ for all $k = 0, \dots, n-1$. As we can see, several instances of the integral kernel of the free propagator are showing up in this expression. Indeed, denoting the kernel of $e^{-isH_0}$ on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by $K_s(x, y)$, which is given by $(2\pi i s)^{-d/2} e^{\frac{i}{2}|x-y|^2/s}$ for any $s > 0$, we find

$$u_t^{n,\varepsilon}(\xi) \ = \ \int_{\mathbb{R}^d} K_{\Delta_{n-1}\tau}(\xi, x_{n-1}) \psi_\varepsilon^\xi(x_{n-1}) \cdots \Big( \int_{\mathbb{R}^d} K_{\Delta_0 \tau}(x_1, x_0) \psi_\varepsilon^\xi(x_0)\, \varphi(x_0)\, dx_0 \Big) \cdots dx_{n-1}$$

and since for any $f \in L^2(\mathbb{R}^d)$, the product $\psi_\varepsilon^\xi f$ is in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and $e^{-isH_0}$ maps into $L^2(\mathbb{R}^d)$ as well as $\varphi \in L^2(\mathbb{R}^d)$, we obtain

$$u_t^{n,\varepsilon}(\xi) \ = \ \Big( e^{-i(\Delta_{n-1}\tau)H_0}\, \psi_\varepsilon^\xi\, e^{-i(\Delta_{n-2}\tau)H_0}\, \psi_\varepsilon^\xi \ \cdots \ e^{-i(\Delta_0 \tau)H_0}\, \psi_\varepsilon^\xi\, \varphi \Big)(\xi)$$

for all $\xi \in \mathbb{R}^d$, where $\psi_\varepsilon^\xi$ denotes multiplication by the function $\psi_\varepsilon^\xi$ itself. Without loss of generality, we can choose $\|\phi\|_\infty = 1$, i.e. $\|\psi_\varepsilon^\xi\|_\infty = 1$ for all $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, and thus it follows from the unitarity of $e^{-isH_0}$

$$\|u_t^{n,\varepsilon}\|_2 \leqslant \|\phi\|_\infty^n \|\varphi\|_2 = \|\varphi\|_2 \qquad \forall \varepsilon > 0\,,\ \forall n \in \mathbb{N}$$

Using the definitions of $u_t^{n,\varepsilon}$ and $u_t^n$, and applying Fatou's Lemma to the families of non-negative functions given by $\big(\xi \mapsto |u_t^{n,\varepsilon}(\xi)|^2\big)_{\varepsilon>0}$ and $\big(\xi \mapsto |u_t^n(\xi)|^2\big)_{n\in\mathbb{N}}$, we find

$$\|U_0(t)\varphi\|_2 \leqslant \liminf_{n\to\infty} \|u_t^n\|_2 \leqslant \liminf_{n\to\infty} \big(\liminf_{\varepsilon\to 0} \|u_t^{n,\varepsilon}\|_2\big) \leqslant \|\varphi\|_2$$

which holds for all $\varphi \in \mathscr{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and therefore proves $U_0(t)\varphi \in L^2(\mathbb{R}^d)$ as well as $\|U_0(t)\| \leqslant 1$. □

For the proof of the next assertion, we follow the reasoning in [12, Corollary 4A], under the slightly modified case of $H_0$ only consisting of the Laplacian, instead of an additional harmonic oscillator term, which is considered there. Moreover, we directly prove the result for $d$ dimensions, instead of restricting to $d$=1.

**Proposition 6** *(Fresnel integral representation of the free propagator). For $t > 0$ and any Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$, it holds*

$$U_0(t)\phi = e^{-itH_0}\phi \tag{2.35}$$

*where $H_0 = -\Delta/2$ and $U_0(t)\phi(\xi) = \mathcal{F}_t\big(x \mapsto \phi(x(0)+\xi)\big)$ for all $\xi \in \mathbb{R}^d$ is the bounded linear map on $\mathscr{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ defined in Proposition 5. Hence, by the bounded extension principle and the denseness of $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, it follows $U_0(t) = e^{-itH_0}$ on the whole space $\mathscr{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.*

***Proof.*** Since on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, the Fourier transform is a bijection onto itself, for $\phi \in \mathcal{S}(\mathbb{R}^d)$ we can choose $\psi \in \mathcal{S}(\mathbb{R}^d)$, such that $\phi = \hat{\psi}$. If $\gamma_0^i \in \mathcal{H}_t$, $i = 1, \dots, d$, are defined as in the proof of Proposition 4, then

$$\phi(x(0)+\xi) = \int_{\mathbb{R}^d} e^{i\sum_{i=1}^d \zeta_i \langle x, \gamma_0^i\rangle_t}\, e^{i\xi\zeta} \psi(\zeta)\, d\zeta = \int_{\mathbb{R}^d} e^{i\langle x, \sum_i \zeta_i \gamma_0^i\rangle_t}\, e^{i\xi\zeta}\psi(\zeta)\, d\zeta$$

The Borel map between $\mathbb{R}^d$ and $Y := \operatorname{span}\{\gamma_0^i\}_{i=1}^d$, given by $F(\zeta) := \sum_{i=1}^d \zeta_i\gamma_0^i$ is injective, and the $i$-th coordinate function of $F^{-1}$ is given by $F^{-1}(y)_i = t^{-1}\langle \gamma_0^i, y\rangle$ for all $y \in Y$, where the factor $t^{-1}$ is due to $\langle \gamma_0^i, \gamma_0^j\rangle_t = t\,\delta_{ij}$. Hence

$$\phi(x(0)+\xi) = \int_Y e^{i\langle x,y\rangle_t} e^{i\xi F^{-1}(y)} \psi(F^{-1}(y))\, d(\lambda \circ F^{-1})(y) =: \int_Y e^{i\langle x,y\rangle_t} d\mu(y)$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$. Extending the complex measure $\mu$ to the whole $\mathcal{H}_t$ by 0, allows us to apply Theorem 2 to the right-hand side of (2.35):

$$\begin{aligned}\mathcal{F}_t\big(x \mapsto \phi(x(0)+\xi)\big) &= \int_Y e^{-\frac{i}{2}\|y\|_t^2} e^{i\xi F^{-1}(y)} \psi(F^{-1}(y))\, d(\lambda \circ F^{-1})(y) \\ &= \int_{\mathbb{R}^d} e^{-\frac{i}{2}\|F(\zeta)\|_t^2}\, e^{i\xi\zeta} \psi(\zeta)\, d\zeta = \int_{\mathbb{R}^d} e^{i\xi\zeta} e^{-\frac{it}{2}|\zeta|^2} \psi(\zeta)\, d\zeta\end{aligned}$$

Thus, if we denote the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ by $\mathcal{F}$, then this shows

$$\mathcal{F}_t\big(x \mapsto \phi(x(0)+\xi)\big) \;=\; \Big(\mathcal{F} \circ \mathcal{M}_{\exp(-\frac{it}{2}|\cdot|^2)} \circ \mathcal{F}^{-1}\phi\Big)(\xi) \;=\; (e^{-itH_0}\phi)(\xi)$$

for all $\xi \in \mathbb{R}^d$, where $\mathcal{M}_{\exp(-\frac{it}{2}|\cdot|^2)}$ denotes multiplication by the function $e^{-\frac{it}{2}|\cdot|^2}$. □

The following Proposition forms another key ingredient for the proof of Theorem 3, but has not been proven rigorously in the literature. The fact that Proposition 5 is the special case of Proposition 7 with $V{=}0$, allows us to extend our ideas from there.

**Proposition 7** *($L^2$-boundedness of the FFPI). For $V \in \mathscr{F}(\mathbb{R}^d)$ and any $t > 0$,*

$$\big(U(t)\varphi\big)(\xi) \;=\; \mathcal{F}_t\Big(x \mapsto e^{-i\int_0^t V(x(s)+\xi)ds}\, \varphi(x(0)+\xi)\Big) \tag{2.36}$$

*forms a bounded linear map from $\mathscr{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, with $\|U(t)\| \leqslant e^{t\|\mu\|}$, where $\mu \in \mathcal{M}(\mathbb{R}^d)$, s.th. $\hat{\mu} = V$.*

***Proof.*** As a consequence of being a Banach algebra, $\mathscr{F}(\mathcal{H}_t)$ is closed under pointwise exponentiation, i.e. for $f \in \mathscr{F}(\mathcal{H}_t)$, also $x \mapsto \exp(f(x))$ belongs to $\mathscr{F}(\mathcal{H}_t)$ and together with Proposition 4, we see that the argument of $\mathcal{F}_t$ in (2.36), is a product of Fresnel class functions and therefore an element of $\mathscr{F}(\mathcal{H}_t)$ by itself.

As in the proof of Proposition 5, we can express $U(t)\varphi$ as the pointwise limit of a sequence $\big(u_t^n\big)_{n\in\mathbb{N}}$ of finite-dimensional Fresnel integrals on the spaces $\mathcal{H}_t^{\pi_n}$ of piecewise linear functions. Just as in that proof, it follows that for each $n \in \mathbb{N}$ and all $\xi \in \mathbb{R}^d$, $u_t^n(\xi)$ is the limit of

$$u_t^{n,\varepsilon}(\xi) \;=\; \lambda_n^{-1} \int_{(\mathbb{R}^d)^n} \left(\prod_{k=0}^{n-1} e^{\frac{i}{2}|\eta_{k+1}-\eta_k|^2/\Delta_k t}\right) e^{-i\int_0^t V(x_\eta(s)+\xi)ds}\, \phi^{\otimes n}(\varepsilon\eta)\, \varphi(\eta_0+\xi)\, d\eta$$

as $\varepsilon \to 0$, where $x_\eta(s)$ is the piecewise linear function on [0,t] having its nodes only in the points $t_k$, i.e. if we set $\eta_n = 0$, then

$$x_\eta(s) \;=\; \sum_{k=0}^{n-1} \Big(\eta_k + \frac{s-t_k}{t_{k+1}-t_k}(\eta_{k+1}-\eta_k)\Big) \mathbb{1}_{[\tau_k,\tau_{k+1}]}(s)$$

and $\phi \in \mathcal{S}^*(\mathbb{R}^d)$ $(= \mathcal{S}(\mathbb{R}^d) \cap \{\phi \,|\, \phi(0) = 1\})$ with $\|\phi\|_\infty = 1$. Performing the change of variables $x_k := \eta_k + \xi$ for all $k = 0, \dots, n$ and splitting up $\int_0^t V(x_\eta(s)+\xi)ds$ according to the partition of $[0,t]$ consisting of $I_k := [t_k, t_{k+1})$, gives

$$\begin{aligned} u_t^{n,\varepsilon}(\xi) \;=\; & \int_{\mathbb{R}^d} K_{\Delta_{n-1}t}(\xi, x_{n-1})\, e^{-i\int_{I_{n-1}} V(x_{n-1}+\frac{s-t_{n-1}}{t_n-t_{n-1}}(\xi-x_{n-1}))ds}\; \psi_\varepsilon^\xi(x_{n-1}) \\ & \cdots \Big( \int_{\mathbb{R}^d} K_{\Delta_0 t}(x_1, x_0)\, e^{-i\int_{I_0} V(x_0+\frac{s-t_0}{t_1-t_0}(x_1-x_0))ds}\; \psi_\varepsilon^\xi(x_0)\, \varphi(x_0)\, dx_0 \Big) \cdots dx_{n-1} \end{aligned}$$

where $K_\tau(x,y)$ denotes the kernel of $e^{-i\tau H_0}$ on $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, and $\psi_\varepsilon^\xi := \phi(\varepsilon(\cdot-\xi))$. In order to proceed, we observe the following: *For any fixed $\tau\in\mathbb{R}$, $u,t\in\mathbb{R}_+$ with $u<t$, and $I:=[u,t]$,*

$$(Tf)(x) := \int_{\mathbb{R}^d} K_\tau(x,y)\, e^{-i\int_I V(y+\frac{s-u}{t-u}(x-y))ds}\, f(y)\, dy$$

*defines a bounded linear map $T : L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|T\| \leqslant e^{|t-u|\|\mu\|}$, where $\mu\in\mathcal{M}(\mathbb{R}^d)$, s.th. $\hat{\mu}=V$.* Indeed, since

$$\Big|\sum_{k=0}^N \frac{(-i)^k}{k!} K_\tau(x,y) \int_I \cdots \int_I V\big(y+\frac{s_1-u}{t-u}\,(x-y)\big)\cdots V\big(y+\frac{s_k-u}{t-u}\,(x-y)\big) ds_1\cdots ds_k\, f(y)\Big|$$

is bounded from above by $(2\pi|\tau|)^{-d/2} e^{\|V\|_\infty |t-u|}|f| \in L^1(\mathbb{R}^d)$ for all $N\in\mathbb{N}$, we are allowed to apply the theorem of dominated convergence, in order to obtain

$$\begin{aligned} Tf(x) \;=\; & \sum_{k=0}^\infty \frac{(-i)^k}{k!} \int_{\mathbb{R}^d} K_\tau(x,y) \int_I\cdots\int_I \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e^{i\sum_{l=1}^k \big(y+\frac{s_l-u}{t-u}(x-y)\big)\cdot\alpha_l} \times \\ & \times f(y)\, d\mu(\alpha_1)\cdots d\mu(\alpha_k)\, ds_1\cdots ds_k\, dy \end{aligned}$$

Since $|\mu|(\mathbb{R}^d)<\infty$, $|I|<\infty$ and $f\in L^1(\mathbb{R}^d)$, by applying Fubini's theorem, we obtain

$$\begin{aligned} Tf(x) \;=\; & \sum_{k=0}^\infty \frac{(-i)^k}{k!} \int_I\cdots\int_I \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e^{i\big(\sum_{l=1}^k \frac{s_l-u}{t-u}\alpha_l\big)\cdot x} \times \\ & \times \Big( \int_{\mathbb{R}^d} K_\tau(x,y)\, e^{i\,\big(\sum_{l=1}^k \frac{t-s_l}{t-u}\alpha_l\big)\cdot y}\, f(y)\, dy\Big)\, d\mu(\alpha_1)\cdots d\mu(\alpha_k)\, ds_1\cdots ds_k \\ =\; & \sum_{k=0}^\infty \frac{(-i)^k}{k!} \int_I\cdots\int_I \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e^{i\big(\sum_{l=1}^k \frac{s_l-u}{t-u}\alpha_l\big)\cdot x} \times \\ & \times \Big[ e^{-i\tau H_0}\Big( e^{i\big(\sum_{l=1}^k \frac{t-s_l}{t-u}\alpha_l\big)\cdot(\cdot)} f\Big)\Big](x)\;\, d\mu(\alpha_1)\cdots d\mu(\alpha_k)\, ds_1\cdots ds_k \end{aligned}$$

Hence, by using the notation $\alpha=(\alpha_1,\ldots,\alpha_k)\in(\mathbb{R}^d)^k$ and $s=(s_1,\ldots,s_k)\in I^k$,

$$|Tf(x)| \leqslant \sum_k \frac{1}{k!} \int_{I^k}\int_{(\mathbb{R}^d)^k} \Big|\Big[ e^{-i\tau H_0}\Big( e^{i\big(\sum_{l=1}^k \frac{t-s_l}{t-u}\alpha_l\big)\cdot(\cdot)} f\Big)\Big](x)\Big|\, d|\mu|^{\otimes k}(\alpha)\, d\lambda^k(s)$$

where $\lambda^k$ denotes the Lebesgue measure in $\mathbb{R}^k$. Let $g_{\alpha,s} := \big|e^{-i\tau H_0}\big(e^{i(\sum_{l=1}^k \frac{t-s_l}{t-u}\alpha_l)\cdot(\cdot)} f\big)\big|$, which satisfies $\|g_{\alpha,s}\|_\infty \leqslant (2\pi\tau)^{-d/2}\|f\|_1$, as well as $\|g_{\alpha,s}\|_2 = \|f\|_2$. Another application of Fubini's theorem, in this case to the counting measure on $\mathbb{N}$, gives

$$
\begin{aligned}
\|Tf\|_2^2 &\leqslant \int_{\mathbb{R}^d} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \int_{I^k} \int_{(\mathbb{R}^d)^k} g_{\alpha,s}(x)\, d|\mu|^{\otimes k}(\alpha)\, d\lambda^k(s) \right)^2 dx \\
&= \sum_{k,m=0}^{\infty} \frac{1}{k!\,m!} \int_{\mathbb{R}^d} \int_{I^k} \int_{(\mathbb{R}^d)^k} g_{\alpha,s}(x)\, d|\mu|^{\otimes k}(\alpha)\, d\lambda^k(s)\ \times \\
&\qquad\qquad \times \int_{I^m} \int_{(\mathbb{R}^d)^m} g_{\alpha',s'}(x)\, d|\mu|^{\otimes m}(\alpha')\, d\lambda^m(s')\, dx \\
&= \sum_{k,m=0}^{\infty} \frac{1}{k!\,m!} \int_{I^k} \int_{I^m} \int_{(\mathbb{R}^d)^k} \int_{(\mathbb{R}^d)^m} \Big( \int_{\mathbb{R}^d} g_{\alpha,s}(x) g_{\alpha',s'}(x)\, dx \Big) \\
&\qquad\qquad d|\mu|^{\otimes m}(\alpha')\, d|\mu|^{\otimes k}(\alpha)\, d\lambda^m(s')\, d\lambda^k(s) \\
&\overset{\text{C.S.}}{\leqslant} \sum_{k,m=0}^{\infty} \frac{(|I|\,\|\mu\|)^{k+m}}{k!\,m!} \|f\|_2^2 \;=\; e^{2(t-u)\|\mu\|}\, \|f\|_2^2
\end{aligned}
$$

This proves both, $Tf \in L^2(\mathbb{R}^d)$ for any $f \in L^1 \cap L^2$, and $\|T\| \leqslant e^{|t-u|\|\mu\|}$. By applying this result $n$ times to the above expresson for $u_t^{n,\varepsilon}(\xi)$, we find

$$
\|u_t^{n,\varepsilon}\|_2 \;\leqslant\; e^{t_1\|\mu\|} e^{(t_2-t_1)\|\mu\|} \cdots e^{(t-t_{n-1})\|\mu\|} \|\varphi\|_2 \;=\; e^{t\|\mu\|}\|\varphi\|_2
$$

where we used that $\psi_\varepsilon^\xi f$ is in $L^1 \cap L^2$ for any $f \in L^2$, and $\|\psi_\varepsilon^\xi\|_\infty = \|\phi\|_\infty = 1$. Hence, by Fatou, $\|U(t)\varphi\|_2 \leqslant \liminf_{n\to\infty} \liminf_{\varepsilon\to 0} \|u_t^{n,\varepsilon}\|_2 \leqslant e^{t\|\mu\|}\|\varphi\|_2$, for any $\varphi \in \mathcal{F} \cap L^2$. Thus, for each $t > 0$, $U(t)$ maps into $L^2$ and is bounded, with $\|U(t)\| \leqslant e^{t\|\mu\|}$. □

Next, let us switch to a different topic, which we will also need the proof below. Adopting the notation from [35, 4.1], let $S(I, L^2)$ denote the linear space of $L^2(\mathbb{R}^d)$-valued simple functions on a given compact intervall $I \subset \mathbb{R}$, where as usual, a function is called simple, if its image is finite, $f(I) = \{h_i\}_{i=1}^n \subset L^2(\mathbb{R}^d)$, and the sets $f^{-1}(h_i)$ are Borel measurable. Moreover, let $R(I, L^2)$, the space of *regulated functions*, be the completion of $S(I, L^2)$ with respect to the norm $\|f\|_\infty := \sup_{t\in I} \|f(t)\|_2$.

On $R(I, L^2)$, the $L^2$-valued Riemann integral is defined by using the bounded extension principle on the integral of simple functions [35, 4.1]. We have the following simple result.

**Proposition 8** *(pointwise evaluation of $L^2$-valued Riemann integrals). If $\psi \in R(I, L^2)$, then for a.e. $\xi \in \mathbb{R}^d$, it holds*

$$
\left( \int_I \psi(t)\, dt \right)(\xi) \;=\; \int_I \big(\psi(t)\big)(\xi)\, dt \tag{2.37}
$$

***Proof.*** In the case when $\psi$ is simple, say $f = \sum_{k=1}^n h_k \mathbb{1}_{I_k}$ with $\{h_k\}_{k=1}^n \subset L^2(\mathbb{R}^d)$, the statement follows directly from the definition of its integral, $\int_I f(t)dt = \sum_{i=1}^k |I_k|\, h_k$.

For the general case of $\psi \in R(I, L^2)$, choose a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions approximating $\psi$ in $R(I, L^2)$, i.e. $\|\psi - f_n\|_\infty \to 0$ as $n \to \infty$. Then $\int_I \psi(t)dt$ is (by definition) the $L^2$-limit of $\int_I f_n(t)dt$. Due to the Riesz-Fischer theorem, we can find a subsequence $(f_{n_m})_{m\in\mathbb{N}}$, such that $\big(\int_I f_{n_m}(t)dt\big)_m$ converges also pointwise, i.e. for a.e. $\xi \in \mathbb{R}^d$,

$$\lim_{m\to\infty} \left(\int_I f_{n_m}(t)\, dt\right)(\xi) \;=\; \left(\int_I \psi(t)\, dt\right)(\xi)$$

By assumption, also $\|f_{n_m}(t) - \psi(t)\|_2 \to 0$ as $m \to \infty$, uniformly in $t$. Hence, another application of Riesz-Fischer allows us to choose a subsequence $(\psi_k)_{k\in\mathbb{N}} \subset (f_{n_m})_{m\in\mathbb{N}}$, such that for all $t \in I$, $\big(\psi_k(t)\big)(\xi) \to \big(\psi(t)\big)(\xi)$ as $k \to \infty$, for a.e. $\xi \in \mathbb{R}^d$. Finally,

$$\left(\int_I \psi(t)\, dt\right)(\xi) \;=\; \lim_{k\to\infty} \left\{\Big(\int_I \psi_k(t)\, dt\Big)(\xi)\right\} \;=\; \lim_{k\to\infty} \int_I \big(\psi_k(t)\big)(\xi)\, dt \;=\; \int_I \big(\psi(t)\big)(\xi)\, dt$$

for a.e. $\xi \in \mathbb{R}^d$, where the last equality follows from the fact that $\big(\psi_k(\cdot)\big)(\xi)$ forms a sequence in $S(I, \mathbb{C})$ converging pointwise to $\big(\psi(t)\big)(\xi)$. □

In the literature on the Fresnel integral and its application to the Feynman path integral, many authors like to refer to the Dyson expansion of unitary propagators given in [28, Theorem X.69]. But the problem is, that the Dyson expansion given there cannot be directly applied, since it has a slightly different structure than it is needed. Even though, it might be possible to somehow derive the needed expansion from there, the author of this thesis couldn't find a way to do so. Hence, we just prove the required statement from scratch.

**Proposition 9** *(a Dyson-type expansion). For any $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ and $t \geqslant 0$, let $V(t)$ denote the composition $e^{itH_0} V e^{-itH_0}$ on $L^2(\mathbb{R}^d)$, and for all $s, t \geqslant 0$ define a sequence $(A_n(t,s))_{n\in\mathbb{N}}$ of bounded linear maps by*

$$A_1(t,s)\varphi \;:=\; \varphi - i \int_s^t V(t_1)\varphi\, dt_1\,,$$

$$A_n(t,s)\varphi \;:=\; A_1(t,s)\varphi + \sum_{k=2}^n (-i)^k \int_s^t \int_{t_k}^t \cdots \int_{t_2}^t V(t_1)\cdots V(t_k)\varphi\, dt_1 \cdots dt_k \quad (n \geqslant 2)$$

*where the integrals are understood as Riemann integrals of Banach space-valued functions on the real line, as considered in Proposition 8.*

*Then the sequence $(A_n(t,s))_{n\in\mathbb{N}}$ is convergent in operator norm topology, and its limit $A(t,s)$ defines a two-parameter family $(A(t,s))_{s,t\geqslant 0}$ of unitary operators on $L^2(\mathbb{R}^d)$, which are jointly strongly continuous and satisfy $A(r,s)A(s,t) = A(r,t)$ as well as $A(t,t) = \mathbb{1}$ for all $r, s, t \geqslant 0$, which means that $(A(t,s))_{s,t\geqslant 0}$ forms a unitary propagator on $L^2(\mathbb{R}^d)$.*

*Moreover, if we set $A(t) := A(t,0)$, then for all $t \geqslant 0$ and any $\varphi \in L^2(\mathbb{R}^d)$,*

$$\frac{d}{dt}A(t)\varphi \;=\; -i\,V(t)A(t)\varphi \tag{2.38}$$

*where $\frac{df}{dt}(t) := \lim_{h\to 0}\frac{1}{h}\big(f(t+h)-f(t)\big)$ denotes the strong derivative of an $L^2$-valued function $t \mapsto f(t)$, whenever the limit exists.*

***Proof.*** *Step 1 (Convergence)* – For any measurable $L^2(\mathbb{R}^d)$-valued function $h$ on $\mathbb{R}$ and $a, b \in \mathbb{R}$ with $a \leqslant b$, it holds $\| \int_a^b h(t)\,dt\|_2 \leqslant \int_a^b \|h(t)\|_2\,dt$ [35, p.113] and thus, from the unitarity of $e^{-itH_0}$ for all $t \in \mathbb{R}$, it follows

$$\big\|A_n(t,s)\varphi - A_m(t,s)\varphi\big\|_2 \;\leqslant\; \sum_{k=m+1}^{n} \frac{|t-s|^k\,\|V\|_\infty^k}{k!}\;\|\varphi\|_2$$

for all $n, k \in \mathbb{N}$, $s, t \geqslant 0$ and $\varphi \in L^2(\mathbb{R}^d)$. Hence $A_n(t,s) - A_m(t,s)$ is norm bounded from above by $\sum_{k=m+1}^n |t-s|^k\|V\|_\infty^k/k!$, which shows that $(A_n(t,s))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathcal{L}(L^2(\mathbb{R}^d)), \|\cdot\|)$ of bounded linear maps from $L^2(\mathbb{R}^d)$ to itself, and thus converges to some $A(t,s) \in \mathcal{L}(L^2)$.

*Step 2 (Continuity)* – Strong continuity of $A(t,s)$ reduces to strong continuity of $A_n(t,s)$ for each $n \in \mathbb{N}$, since it is preserved under operator norm convergence. Let us first show, that *for a given $\varphi \in L^2(\mathbb{R}^d)$ and a map $T : \mathbb{R}\times\mathbb{R} \to \mathcal{L}(L^2)$, for which both $s \mapsto T(t_0, s)$ and $t \mapsto T(t, s_0)$ are strongly continuous whenever $t_0, s_0 \in \mathbb{R}$ are fixed, the map*

$$I_u : \mathbb{R} \to L^2(\mathbb{R}^d), t \mapsto \int_u^t T(t,s)\varphi\,ds$$

*is continuous for any $u \in \mathbb{R}$.* For this let $\varepsilon > 0$ and without loss of generality assume $t_0 > u$. We are free to choose an arbitrary constant $R > t_0$ and to restrict $t$ to vary in $[u, R]$. If $T(s,t)\varphi = 0$ for all $s, t \in [u, R]$, then nothing is to be shown, hence we assume $T(s,t)\varphi \neq 0$ for at least one combination of $s, t \in [u, R]$. By continuity of $t \mapsto T(t,s)\varphi$, for each $s \in [u, R]$ we can find $\delta(s) > 0$ such that $\|T(t,s)\varphi - T(t_0,s)\varphi\|_2 < \frac{\varepsilon}{2|R-u|}$ whenever $|t - t_0| < \delta(s)$. Setting

$$\delta_1 := \min_{s\in[u,R]} \delta(s)\;, \quad \delta_2 := \Big(2 \max_{s,t\in[u,R]} \|T(t,s)\varphi\|_2\Big)^{-1}\varepsilon$$

where $\delta_2$ is well-defined by assumption, we obtain

$$\begin{aligned}\big\|I_u(t) - I_u(t_0)\big\|_2 &\leqslant \Big\| \int_{t_0}^{t} T(t,s)\varphi\, ds\Big\|_2 + \Big\| \int_{u}^{t_0} \Big(T(t,s)\varphi - T(t_0,s)\varphi\Big)\, ds\Big\|_2 \\ &\leqslant |t-t_0| \max_{s\in[u,R]} \big\|T(t,s)\varphi\big\|_2 + |t_0-u| \max_{s\in[u,t_0]} \big\|T(t,s)\varphi - T(t_0,s)\varphi\big\|_2 \\ &< \delta_2 \max_{s,t\in[u,R]} \big\|T(t,s)\varphi\big\|_2 + \frac{\varepsilon}{2}\,\frac{|t_0-u|}{|R-u|} < \varepsilon\end{aligned}$$

whenever $|t-t_0| < \delta := \min\{\delta_1,\delta_2\}$. This shows the continuity of $I_u$ for all $u\in\mathbb{R}$. Since $e^{-isH_0}$ and thus also $V(s)$ are strongly continuous for all $s\in\mathbb{R}$, it's a direct application of this result, that $A_1(t,s)$ is separately strongly continuous, where in this particular case $T(t,s) = V(s)$ for all $t\in\mathbb{R}$. Moreover, each of the iterated integrals in the $k$th term of $A_n(t,s)\varphi$ can be written in the form of $I_u$, where, when seen from the $t$ variable, $u$ plays either the role of $s$ or of one of the $t_j$, and when seen from the respective integration variable, then $u$ is given by $t$. Thus each term in $A_n(t,s)\varphi$ is separately continuous in $s$ and $t$, and so is $A_n(t,s)\varphi$ for each $n\in\mathbb{N}$ by itself, which in turn translates to *separate* strong continuity of $A(t,s)$. For any $S,T>0$, it holds

$$\|A(t,s)\| \leqslant \sum_{k=0}^{\infty} \frac{|t-s|^k \|V\|_\infty^k}{k!} = e^{|t-s|\|V\|_\infty} \leqslant e^{(S+T)\|V\|_\infty}$$

for all $(s,t)\in[-S,S]\times[-T,T]$, which shows that $A(t,s)$ is locally uniformly bounded. As was proved in [16, Theorem 2.2], given that the rest of the properties of a unitary propagator are fullfilled, then *joint* strong continuity is equivalent to separate strong continuity together with local uniform boundedness.

*Step 3* – Next, let us show that for all $s,t\geqslant 0$, it holds $A(s,t) = A(t,s)^*$. Due to its boundedness, we have $A(t,s) = A(t,s)^{**}$ and therefore it is enough to consider the case $s\leqslant t$. For this, we use the property [35, (4.10)] of Riemann integrals of Banach space-valued functions $f:\mathbb{R}\to X$, that for any linear functional $l\in X^*$, we have $l(\int_a^b f(t)dt) = \int_a^b l(f(t))dt$. Applied to $l = \langle\varphi,\cdot\rangle$, we get

$$\begin{aligned}\big\langle A_1(s,t)\psi,\varphi\big\rangle &= \langle\psi,\varphi\rangle + i\int_t^s \big\langle V(t_1)\psi,\varphi\big\rangle\, dt_1 = \langle\psi,\varphi\rangle - i\int_s^t \big\langle \psi, V(t_1)\varphi\big\rangle\, dt_1 \\ \big\langle A_n(s,t)\psi,\varphi\big\rangle &= \big\langle A_1(s,t)\psi,\varphi\big\rangle + \sum_{k=2}^{n} i^k \int_t^s\int_{t_k}^s \cdots \int_{t_2}^s \big\langle V(t_1)\cdots V(t_k)\psi,\varphi\big\rangle\, dt_1\cdots dt_k \\ &= \big\langle \psi, A_1(t,s)\varphi\big\rangle + \sum_{k=2}^{n} (-i)^k \int_s^t\int_s^{t_k} \cdots \int_s^{t_2} \big\langle \psi, V(t_k)\cdots V(t_1)\varphi\big\rangle\, dt_1\cdots dt_k\end{aligned}$$

where we used that $V$ is real valued and therefore $V(t)$ is symmetric. If we *relabel* the integration variables $t_1,\dots,t_k$ in the $k$th term of the last expression to $t_k,\dots,t_1$, and if we use the identity $\mathbb{1}_{[s,t_j]}(t_{j+1}) = \mathbb{1}_{[t_{j+1},t]}(t_j)$, then we obtain for the $k$th term

$$\begin{aligned}
&(-i)^k \int_s^t \cdots \int_s^t \mathbb{1}_{[s,t_1]}(t_2) \cdots \mathbb{1}_{[s,t_{k-1}]}(t_k) \left\langle \psi, V(t_1) \cdots V(t_k)\varphi \right\rangle \, dt_k \cdots dt_1 \\
&= (-i)^k \int_s^t \cdots \int_s^t \mathbb{1}_{[t_2,t]}(t_1) \cdots \mathbb{1}_{[t_k,t]}(t_{k-1}) \left\langle \psi, V(t_1) \cdots V(t_k)\varphi \right\rangle \, dt_k \cdots dt_1 \\
&= (-i)^k \int_s^t \int_{t_k}^t \cdots \int_{t_2}^t \left\langle \psi, V(t_1) \cdots V(t_k)\varphi \right\rangle \, dt_1 \cdots dt_k
\end{aligned}$$

Therefore, we find $\langle A_n(s,t)\psi, \varphi\rangle = \langle \psi, A_n(t,s)\varphi\rangle$ for all $\psi, \varphi \in L^2(\mathbb{R}^n)$ and $n \in \mathbb{N}$. By continuity of the inner product, the same holds true for $A(t,s)$, hence boundedness of $A(t,s)$ allows the conclusion $A(t,s)^* = A(s,t)$ for all $s, t \geqslant 0$.

*Step 4* – Now, in order to prove $A(t,s)A(s,r) = A(t,r)$, for any $r, s, t \geqslant 0$, let us first show, that *for $r \leqslant s \leqslant t$ and any $N \in \mathbb{N}$, it holds*

$$A_N(t,s)A_N(s,r) = A_N(t,r) + R_N(t,s,r)$$

*where $R_N(t,s,r)$ is a bounded linear map with $\|R_N(t,s,r)\| \to 0$ as $N \to \infty$.* For all $\varphi, \psi \in L^2(\mathbb{R}^d)$, the inner product $\langle \psi, A_N(t,s)A_N(s,r)\varphi\rangle$ equals

$$\sum_{k,l=0}^{N} (-i)^{k+l} \int_s^t \int_{s_k}^t \cdots \int_{s_2}^t \int_r^s \int_{r_l}^s \cdots \int_{r_2}^s f(s_1, \ldots, s_k, r_1, \ldots, r_l) \, dr_1 \cdots dr_l ds_k \ldots ds_l$$

where $f(s_1, \ldots, s_k, r_1, \ldots, r_l) := \left\langle \psi, \prod_{i=1}^k V(s_i) \prod_{j=1}^l V(r_j)\varphi \right\rangle$ and $k, l = 0$ denote the very first term in the definition of $A_N$. For any family of complex numbers $(\alpha_{k,l})_{(k,l)\in Q_N}$, where $Q_N$ denotes the square $[0,N] \times [0,N]$, it holds

$$\sum_{Q_N} \alpha_{k,l} \;=\; \sum_{T_N} \alpha_{k,l} \;+ \sum_{Q_N \setminus T_N} \alpha_{k,l}$$

and $T_N$ denotes the triangle $\{(k,l) \in Q_N \,|\, k + l \leqslant N\}$. We will show below, that the second term in this splitting, when applied to the sum above, gives rise to the map $R_N(t,s,r)$, more precisely, the second term will be given by $\langle \psi, R_N(t,s,r)\varphi\rangle$.

Strictly speaking, the notation in many of the following steps do only make sense when $k \geqslant 3$, but for smaller $k$ the computations involve less factors and therefore are much simpler. Also, they can easily be reduced from the given ones by neglecting the additional terms which do only exist for bigger $k$.

The sum $\sum_{T_N} \alpha_{k,l}$ can also be written as $\sum_{k=0}^N \sum_{j=0}^k \alpha_{j,k-j}$. Renaming integration variables in each of the iterative integrals, from $(s_1, \ldots, s_j, r_1, \ldots, r_{k-j})$ to $(t_1, \ldots, t_k)$, in other words, for fixed $k$ and $j$, setting $t_i := s_i$ whenever $i \leqslant j$ and $t_i := r_{i-j}$ for $j + 1 \leqslant i \leqslant k$, gives for the sum over $T_N$

$$\begin{aligned}
&\sum_{k=0}^{N} (-i)^k \sum_{j=0}^{k} \int_s^t \int_{t_j}^t \cdots \int_{t_2}^t \int_r^s \int_{t_k}^s \cdots \int_{t_{j+2}}^s f(t_1, \ldots, t_k) \, dt_{j+1} \cdots dt_k dt_1 \cdots dt_j \\
&= \sum_{k=0}^{N} (-i)^k \int_r^t \cdots \int_r^t \zeta^k_{r,s,t}(t_1, \ldots, t_k) \, f(t_1, \ldots, t_k) \, dt_1 \ldots dt_k
\end{aligned} \tag{$*$}$$

where we are allowed to interchange integration order, due to the boundedness of the integrand and the finiteness of the product spaces $[r,t]^k$, and we also have set

$$\zeta^k_{r,s,t}(t_1,\dots,t_k) := \sum_{j=0}^{k} \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[s,t)}(t_j)\mathbb{1}_{[t_{j+2},s)}(t_{j+1})\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k)$$

whenever $k \geqslant 1$, and $\zeta^0_{r,s,t} = 1$ (but as pointed out, for $k=0$, there are no integrals present and the whole expression is just equal to $\langle\psi,\varphi\rangle$). The first two terms in this sum are given by

$$\begin{aligned}
&\mathbb{1}_{[t_2,s)}(t_1)\mathbb{1}_{[t_3,s)}(t_2)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k) + \mathbb{1}_{[s,t)}(t_1)\mathbb{1}_{[t_3,s)}(t_2)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})t_{[r,s)}(t_k)\\
&= \left(\mathbb{1}_{[t_2,s)}(t_1) + \mathbb{1}_{[s,t)}(t_1)\right)\mathbb{1}_{[t_3,s)}(t_2)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})t_{[r,s)}(t_k)\\
&= \mathbb{1}_{[t_2,t)}(t_1)\mathbb{1}_{[t_3,s)}(t_2)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k)
\end{aligned}$$

and thus, we claim that *the first $j$ terms can be combined to*

$$\mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[t_{j+1},s)}(t_j)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k) \qquad (**)$$

*where* $j = 2,\dots,k$. For $j=2$, this is the expression, we just calculated. Assuming, that this holds for some $j$, then the sum of the first $j+1$ terms equals

$$\begin{aligned}
&\mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[t_{j+1},s)}(t_j)\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k)\\
&+ \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[s,t)}(t_j)\mathbb{1}_{[t_{j+2},s)}(t_{j+1})\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k)\\
&= \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[t_{j+1},t)}(t_j)\mathbb{1}_{[t_{j+2},s)}(t_{j+1})\cdots\mathbb{1}_{[t_k,s)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k)
\end{aligned}$$

and thus the claim holds also for the first $j+1$ terms. By induction, this shows that $(**)$ is true for all $j = 2,\dots,k$. Hence, by writing $\zeta^k_{r,s,t}$ as the sum of the $(k+1)$th term and the first $k$ terms, it follows

$$\begin{aligned}
\zeta^k_{r,s,t}(t_1,\dots,t_k) &= \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_k,t)}(t_{k-1})\mathbb{1}_{[r,s)}(t_k) + \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_k,t)}(t_{k-1})\mathbb{1}_{[s,t)}(t_k)\\
&= \mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_k,t)}(t_{k-1})\mathbb{1}_{[r,t)}(t_k)
\end{aligned}$$

Plugging this back into $(*)$ and defining $R_N(t,s,r)$ as proposed above, we obtain

$$\begin{aligned}
&\langle\psi, A_N(t,s)A_N(s,r)\varphi\rangle - \langle\psi, R_N(t,s,r)\varphi\rangle\\
&= \sum_{k=0}^{N}(-i)^k\int_r^t\cdots\int_r^t \zeta^k_{r,s,t}(t_1,\dots,t_k)\, f(t_1,\dots,t_k)\,dt_1\dots dt_k\\
&= \sum_{k=0}^{N}(-i)^k\int_r^t\int_{t_k}^t\cdots\int_{t_2}^t \langle\psi, V(t_1)\cdots V(t_k)\varphi\rangle\,dt_1\cdots dt_k \;=\; \langle\psi, A_N(t,r)\varphi\rangle
\end{aligned}$$

for all $\varphi,\psi \in L^2(\mathbb{R}^d)$. It remains to show, that $R_N(t,s,r)$ converges to 0 in operator norm, as $N \to \infty$. We have

$$\big\|R_N(t,s,r)\varphi\big\|_2 \leqslant \sum_{(k,l)\in Q_N\setminus T_N} \frac{|t-r|^{k+l}\,\|V\|_\infty^{k+l}}{k!\;l!}\,\|\varphi\|_2$$

Since $Q_{\lfloor \frac{N}{2}\rfloor} \subset T_N$, the difference $Q_N \setminus T_N$ is a subset of $Q_N \setminus Q_{\lfloor \frac{N}{2}\rfloor}$, and therefore

$$\sum_{(k,l)\in Q_N\setminus T_N} \frac{|t-r|^{k+l}\,\|V\|_\infty^{k+l}}{k!\;l!} \leqslant \left(\sum_{k=0}^{N}\frac{|t-r|^k\,\|V\|_\infty^k}{k!}\right)^2 - \left(\sum_{k=0}^{\lfloor N/2\rfloor}\frac{|t-r|^k\,\|V\|_\infty^k}{k!}\right)^2$$

The difference on the right-hand side converges to 0 as $N \to \infty$, because the sequence of squared partial sums $\big(\sum_{k=0}^N |t-r|^k\|V\|_\infty^k/k!\big)^2$ is a Cauchy sequence (since it converges to $e^{2|t-r|\|V\|_\infty}$). Thus, we have $\|R_N(t,s,r)\| \to 0$ as $n \to \infty$, which completes the proof of the claim stated at the beginning of this step. Concerning the limit $N \to \infty$, by using that $A_N(t,s)$ is uniformly bounded (in $N$), $\|A_N(t,s)\| \leqslant e^{|t-s|\|V\|_\infty}$ $\forall N \in \mathbb{N}$, and converges in operator norm, we obtain

$$\begin{aligned}
&\big\|A_N(t,s)A_N(s,r) - A(t,s)A(s,r)\big\| \\
&\leqslant \big\|A_N(t,s)\big(A_N(s,r)-A(s,r)\big)\big\| + \big\|\big(A_N(t,s)-A(t,s)\big)A(s,r)\big\| \\
&\leqslant e^{|t-s|\|V\|_\infty}\big\|A_N(s,r)\varphi - A(s,r)\big\| + \big\|A_N(t,s)-A(t,s)\big\|\big\|A(s,r)\big\|
\end{aligned}$$

which converges to 0 as $N \to \infty$. Therefore, from the previous result, it follows for all $r,s,t \in [0,\infty)$ with $r \leqslant s \leqslant t$,

$$A(t,s)A(s,r) = \lim_{N\to\infty} A_N(t,s)A_N(s,r) = \lim_{N\to\infty} A_N(t,r) + \lim_{N\to\infty} R_N(t,s,r) = A(t,r) \qquad (i)$$

Next, let $r = t \geqslant s$, then the computation leading to equation $(*)$ still holds true, but in this case $\zeta^k_{r,s,t}(t_1,\ldots,t_k) =: \zeta^k_{s,t}(t_1,\ldots,t_k)$ is given by

$$\sum_{j=0}^{k}(-1)^{k-j}\mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_j,t)}(t_{j-1})\mathbb{1}_{[s,t)}(t_j)\mathbb{1}_{[s,t_{j+2})}(t_{j+1})\cdots\mathbb{1}_{[s,t_k)}(t_{k-1})\mathbb{1}_{[s,t)}(t_k)$$

and very similar to above, we obtain that the first $k$ terms add up to

$$-\mathbb{1}_{[t_2,t)}(t_1)\cdots\mathbb{1}_{[t_k,t)}(t_{k-1})\mathbb{1}_{[s,t)}(t_k)$$

which, except for the sign, equals exactly the term with $j = k$, and hence $\zeta^k_{s,t} = 0$ for all $k \geqslant 1$. Therefore, in the sum over $T_N$, only the term $k = 0$ survives. This shows

$$\langle\psi, A_N(t,s)A_N(s,t)\varphi\rangle = \langle\psi,\varphi\rangle + \langle\psi, R_N(t,s)\varphi\rangle$$

where $R_N(t,s)$, defined in the same manner as before, also converges to 0 in operator norm (proof is identical), and therefore

$$A(t,s)A(s,t) \;=\; \lim_{N\to\infty} A_N(t,s)A_N(s,t) + \lim_{N\to\infty} R_N(t,s) \;=\; \mathbb{1} \;=\; A(t,t) \qquad (ii)$$

for all $s,t \in [0,\infty)$ with $s \leqslant t$. So far, we have shown the identity $A(t,s)A(s,r) = A(t,r)$ in the cases $t \geqslant s \geqslant r$ and $r = t \geqslant s$ (equation $(i)$ and $(ii)$, respectively). These can be used to prove the result for all the other combinations of $r,s,t \geqslant 0$: Applying step 3 to equation $(i)$ covers the case $r \geqslant s \geqslant t$. By using $A(s,r) = A(s,t)A(t,r)$ and $A(t,s)A(s,t) = \mathbb{1}$ for $r > t \geqslant s$, we get

$$A(t,s)A(s,r) = A(t,s)A(s,t)A(t,r) = A(t,r) \qquad (iii)$$

and another application of step 3 proves the identity for $s \geqslant t \geqslant r$. For the only remaining case, i.e. $t \geqslant r \geqslant s$ (and its reverse), we observe, that interchanging $r$ and $t$ in $(iii)$ gives $A(r,s)A(s,t) = A(r,t)$, which is just the adjoint of $A(t,s)A(s,r) = A(t,r)$.

*Step 5* – Unitarity of $A(t,s)$ is now a simple combination of step 3 and 4, because we have shown $A(t,s) = A(s,t)^*$ and therefore $A(t,s)A(t,s)^* = A(t,t) = \mathbb{1}$, as well as $A(t,s)^*A(t,s) = A(s,s) = \mathbb{1}$, which hold for all $s,t \geqslant 0$.

*Step 6* – It remains to show, that the $L^2$-valued function, given by $f(t) := A(t,0)\varphi$ solves equation (2.38) for any $\varphi \in L^2(\mathbb{R}^d)$ and $t \geqslant 0$. In order to calculate the strong derivative of $A_N(t,0)\varphi$, we claim: *For $k,j \in \mathbb{N}$ with $1 < j \leqslant k \leqslant N$, it holds*

$$\begin{aligned} &\frac{d}{dt}\int_{t_{j+1}}^{t}\int_{t_j}^{t}\cdots\int_{t_2}^{t} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_j \\ &= \int_{t_{j+1}}^{t}\Big(\frac{d}{dt}\int_{t_j}^{t}\cdots\int_{t_2}^{t} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_{j-1}\Big)dt_j \end{aligned} \qquad (*)$$

*for all* $t, t_{j+1}, \ldots, t_k \geqslant 0$ (for notational reasons, $t_{k+1} := 0$). For this purpose, consider the difference quotient

$$\begin{aligned} &\frac{1}{h}\left(\int_{t_{j+1}}^{t+h}\cdots\int_{t_2}^{t+h} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_j \;-\; \int_{t_{j+1}}^{t}\cdots\int_{t_2}^{t} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_j\right) \\ &= \int_{t_{j+1}}^{t} \mathcal{D}_h\left(\int_{t_j}^{t}\cdots\int_{t_2}^{t} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_{j-1}\right)dt_j \;+\; \Phi_h(t_{j+1},\ldots,t_k) \end{aligned}$$

where we use the abbreviation $\mathcal{D}_h f(t) := \frac{1}{h}\big(f(t+h) - f(t)\big)$, and

$$\begin{aligned} \big\|\Phi_h(t_{j+1},\ldots,t_k)\big\|_2 \;&=\; \frac{1}{|h|}\left\|\int_{t}^{t+h}\int_{t_j}^{t+h}\cdots\int_{t_2}^{t+h} V(t_1)\cdots V(t_k)\varphi\,dt_1\cdots dt_{j-1}dt_j\right\|_2 \\ &\leqslant\; \big\|V\big\|_\infty^k\,\big\|\varphi\big\|_2\,\frac{1}{|h|}\int_{t}^{t+|h|}\int_{t_j}^{t+|h|}\cdots\int_{t_2}^{t+|h|} dt_1\cdots dt_{j-1}dt_j \;\leqslant\; \big\|V\big\|_\infty^k\,\big\|\varphi\big\|_2\,|h|^{j-1} \end{aligned}$$

By assumption $j \geqslant 2$, so this shows that $\|\Phi_h(t_{j+1},\ldots,t_k)\|_2 \to 0$ as $h \to 0$. Moreover, forming the inner product of an arbitrary $\psi \in L^2(\mathbb{R}^d)$ with the first term of the above expression for the difference quotient, gives

$$\int_{t_{j+1}}^{t} \mathcal{D}_h \left( \int_{t_j}^{t} \cdots \int_{t_2}^{t} \langle \psi, V(t_1) \cdots V(t_k)\varphi \rangle \, dt_1 \cdots dt_{j-1} \right) dt_j$$

$h$ can be assumed to vary only in a bounded intervall of the form $I = \{s \in \mathbb{R} : |s| \leqslant R\}$ for some $R > 0$. In order to find an $h$-independent upper bound for the integrand, we perform $j - 2$ times the same calculation as above. Then the integrand takes the form

$$\int_{t_j}^{t} \cdots \int_{t_3}^{t} \frac{1}{h} \int_{t}^{t+h} \langle \psi, V(t_1) \cdots V(t_k)\varphi \rangle \, dt_1 \cdots dt_{j-1} \; + \; N_h$$

where $|N_h| \to 0$ as $h \to 0$. Hence it is bounded by $|t|^{j-2} \|V\|^k \|\psi\| \|\varphi\| + \sup_{h \in I} |N_h|$ uniformly in $h \in I$, and therefore, by the theorem of dominated convergence, we are allowed to take the limit $h \to 0$ inside of the outer integral. By continuity of $\langle \psi, \cdot \rangle$, the limit passes also the inner product, and since $\psi \in L^2(\mathbb{R}^d)$ was arbitrary, this completes the proof of $(*)$.

For $j = 1$, by the fundamental theorem of calculus (which holds true for Banach-space valued Riemann integrals [35, p.113]), we find for $1 \leqslant k \leqslant N$ and $t, t_2, \ldots, t_k \geqslant 0$,

$$\frac{d}{dt} \int_{t_2}^{t} V(t_1) \cdots V(t_k)\varphi \, dt_1 \;=\; V(t)V(t_2) \cdots V(t_k)\varphi$$

Next, let us use this together with $(*)$ on each term to find the strong derivative of $A_N(t,0)\varphi$. For $N = 1$, it is given by $-iV(t)\varphi$, and for $N \geqslant 2$,

$$\begin{aligned} \frac{d}{dt} A_N(t,0)\varphi \;&=\; -\,i\,V(t)\varphi \;+\; \int_0^t V(t)V(t_2)\varphi \, dt_2 \\ &\quad + \sum_{2<k\leqslant N} (-i)^k \int_0^t \int_{t_k}^{t} \cdots \int_{t_3}^{t} V(t)V(t_2) \cdots V(t_k)\varphi \, dt_2 \cdots dt_k \qquad (**) \\ &=\; -\,i\,V(t)A_{N-1}(t,0)\varphi \end{aligned}$$

where the notation is such that the sum over $2 < k \leqslant N$ is only non-zero for $N > 2$, and we have used $\int_a^b Ah(t)dt = A \int_a^b h(t)dt$ for any bounded linear map $A$, which (again) can be seen by forming the inner product with an arbitrary $\psi \in L^2(\mathbb{R}^d)$,

$$\left\langle \psi, \int_a^b Ah(t)dt \right\rangle = \int_a^b \langle A^*\psi, h(t) \rangle \, dt = \left\langle A^*\psi, \int_a^b h(t)dt \right\rangle = \left\langle \psi, A \int_a^b h(t)dt \right\rangle$$

Now, from expression $(**)$, it immediately follows that $\frac{d}{dt} A_N(t,0)\varphi$ *converges locally uniformly in* $t$ *to* $-iV(t)A(t,0)\varphi$*, as* $N \to \infty$. Indeed,

$$\Big\| V(t)A_{N-1}(t,0)\varphi - V(t)A(t,0)\varphi \Big\|_2 \;\leqslant\; \|V\|_\infty \sum_{k=N}^{\infty} \frac{\|V\|_\infty^k \, t^k}{k!} \|\varphi\|_2$$

and the exponential series is locally uniformly convergent. This also implies, that for any $\psi \in L^2(\mathbb{R}^d)$, the sequence given by $\langle \psi, \frac{d}{dt} A_N(t,0)\varphi \rangle = \frac{d}{dt}\langle \psi, A_N(t,0)\varphi\rangle$ converges locally uniformly. Therefore, by the theorem of term-by-term differentiation of locally uniformly convergent series [19, p. 333], $\frac{d}{dt}\langle \psi, A(t,0)\varphi\rangle$ is the limit of $\frac{d}{dt}\big\langle \psi, A_N(t,0)\varphi\big\rangle$ as $N \to \infty$, hence

$$\Big\langle \psi, \frac{d}{dt} A(t,0)\varphi \Big\rangle \;=\; \lim_{N\to\infty} \big\langle \psi, -iV(t) A_{N-1}(t,0)\varphi \big\rangle \;=\; \big\langle \psi, -iV(t) A(t,0)\varphi \big\rangle$$

Since $\psi \in L^2(\mathbb{R}^d)$ was arbitrary, this completes the proof of $A(t)\varphi$ being a strong solution to (2.38). □

For the application of Proposition 9 in the proof of Theorem 3, it is required that $A(t)$ leaves the domain of $H$, which is given by $H^2(\mathbb{R}^d)$, invariant. This will be shown in the Lemma below, which couldn't be found in the literature either.

**Lemma 6**. *Let $V \in C^2(\mathbb{R}^d)$ s.th. $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$, for all multi-indices $\alpha \in \mathbb{N}^d$, with $|\alpha| \leqslant 2$, then for any $t \geqslant 0$, the unitary operator $A(t)$, introduced in Proposition 9, leaves $H^2(\mathbb{R}^d)$ invariant.*

***Proof.*** By the Leibniz rule for weak derivatives, $\varphi \in H^2(\mathbb{R}^d)$, $V \in C^2(\mathbb{R}^d)$ as well as $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$ for $|\alpha| \leqslant 2$ imply $V\varphi \in H^2(\mathbb{R}^d)$.

In order to see that $V(t)\varphi$ is in $H^2(\mathbb{R}^d)$, where $V(t) = e^{itH_0} V e^{-itH_0}$, we could just use the fact that $e^{\pm itH_0}\mathcal{D}(H_0) = \mathcal{D}(H_0) = H^2(\mathbb{R}^d)$ [35, 5.1], but for the proof of $A(t)\varphi \in H^2(\mathbb{R}^d)$ we need an explicit estimate on $\|D^\alpha(V(t)\varphi)\|$, $D$ denoting the weak derivative. For this, we first observe that $\mathcal{F}(V(t)\varphi) = e^{\frac{it}{2}|\cdot|^2}\mathcal{F}(V\varphi_0(t))$, where $\varphi_0(t) := e^{-itH_0}\varphi$ and $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^d)$. Indeed, $\mathcal{F}\mathcal{F} = (2\pi)^{-d}\,\mathcal{R}$, and therefore also $\mathcal{F}^{-1}\mathcal{F}^{-1} = (2\pi)^d\,\mathcal{R}$, where $\mathcal{R}$ denotes the reflection $\mathcal{R}f(x) := f(-x)$, which satisfies $\mathcal{R}\mathcal{R} = \mathbb{1}$ as well as $[\mathcal{R}, e^{\frac{it}{2}|\cdot|^2}] = 0$. Hence

$$\mathcal{F} e^{itH_0} \mathcal{F}^{-1} \;=\; \mathcal{F}\mathcal{F}\, e^{\frac{it}{2}|\cdot|^2} \mathcal{F}^{-1}\mathcal{F}^{-1} \;=\; \mathcal{R}\, e^{\frac{it}{2}|\cdot|^2}\mathcal{R} \;=\; e^{\frac{it}{2}|\cdot|^2}$$

From the Fourier characterization of $H^2(\mathbb{R}^d)$, it follows

$$\|D^\alpha(V(t)\varphi)\|_2 \;=\; \|k^\alpha \mathcal{F}(V(t)\varphi)\|_2 \;=\; \|k^\alpha \mathcal{F}(V\varphi_0(t))\|_2 \;=\; \|D^\alpha(V\varphi_0(t))\|$$

The Leibniz rule implies $\|D^\alpha(V\varphi_0(t))\|_2 \;\leqslant\; \sum_{\beta, |\beta|\leqslant|\alpha|} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}V\|_\infty \, \|D^\beta \varphi_0(t)\|_2$, and by performing the same steps as above, $\|D^\alpha\varphi_0(t)\|_2 = \|k^\alpha\hat{\varphi}\|_2 = \|D^\alpha\varphi\|_2$. Hence, by setting $C_V := \max_{|\alpha|\leqslant 2}\|D^\alpha V\|_\infty$, and from $\sum_{|\beta|\leqslant|\alpha|}\binom{\alpha}{\beta} = 2^{|\alpha|}$, we obtain for all $t \geqslant 0$

$$\|D^\alpha(V(t)\varphi)\|_2 \;\leqslant\; 4\,C_V \max_{|\beta|\leqslant 2}\|D^\beta\varphi\|_2 \tag{$*$}$$

Since for all $|\alpha| \leqslant 2$ it holds $\|D^\alpha \varphi\|_2 \leqslant \|\varphi\|_{H^2} < \infty$, $(*)$ implies for any $k \in \mathbb{N}$

$$\|D^\alpha (V(t_1) \cdots V(t_k)\varphi)\|_2 \;\leqslant\; 4^k\, C_V^k\, \|\varphi\|_{H^2(\mathbb{R}^d)}$$

By using the variational characterization of the $L^2$ norm, we find

$$\begin{aligned}
&\Big\| D^\alpha \int_0^t \int_{t_k}^t \cdots \int_{t_2}^t V(t_1)\cdots V(t_k)\varphi \, dt_1 \cdots dt_k \Big\|_2 \\
&= \sup_{g \in C_0^\infty, \|g\|_2 = 1} \Big| \Big\langle \int_0^t \int_{t_k}^t \cdots \int_{t_2}^t V(t_1)\cdots V(t_k)\varphi \, dt_1 \cdots dt_k, \partial^\alpha g \Big\rangle \Big| \\
&= \sup_{g \in C_0^\infty, \|g\|_2 = 1} \Big| \int_0^t \int_{t_k}^t \cdots \int_{t_2}^t \langle V(t_1)\cdots V(t_k)\varphi, \partial^\alpha g\rangle \, dt_1 \cdots dt_k \Big| \\
&\leqslant \sup_{g \in C_0^\infty, \|g\|_2 = 1} \int_0^t \int_{t_k}^t \cdots \int_{t_2}^t |\langle D^\alpha V(t_1)\cdots V(t_k)\varphi, g\rangle| \, dt_1 \cdots dt_k \\
&\overset{\text{C.S.}}{\leqslant} \int_0^t \int_{t_k}^t \cdots \int_{t_2}^t \|D^\alpha V(t_1)\cdots V(t_k)\varphi\|_2 \; dt_1 \cdots dt_k \;\leqslant\; 4^k C_V^k \, \frac{t^k}{k!} \, \|\varphi\|_{H^2}
\end{aligned}$$

Let $(A_N(t))_{N\in\mathbb{N}}$ be the defining sequence of unitary operators, approximating $A(t)$, as introduced in Proposition 9. The previous calculation shows, that $A_N(t)\varphi$ belongs to $H^2(\mathbb{R}^d)$, and moreover $\|D^\alpha A_N(t)\varphi\|_2$ is uniformly bounded in $N$. Explicitly,

$$\|D^\alpha A_N(t)\varphi\|_2 \;\leqslant\; \sum_{k=0}^N \frac{4^k C_V^k t^k}{k!} \|\varphi\|_{H^2} \;\leqslant\; e^{4C_V t}\|\varphi\|_{H^2}$$

which holds for all $N \in \mathbb{N}$. Therefore, from $|\langle A(t)\varphi, \partial^\alpha g\rangle| = \lim_{n\to\infty} |\langle D^\alpha A_N(t)\varphi, g\rangle|$ and $|\langle D^\alpha A_N(t)\varphi, g\rangle| \leqslant e^{4C_V t}\|\varphi\|_{H^2}$ for $\|g\|_2 = 1$, it follows

$$\|D^\alpha A(t)\varphi\|_2 = \sup_{f \in C_0^\infty, \|f\|=1} |\langle A(t), \partial^\alpha g\rangle| \;\leqslant\; e^{4C_V t}\|\varphi\|_{H_2} \;<\; \infty$$

whence $A(t)\varphi \in H^2(\mathbb{R}^d)$, for all $t \geqslant 0$. □

This finishes our preparations for the proof of Theorem 3. Even though we will follow a strategy similar to the one used in [12, 4B], we will come up with a lot of added explicit calculations and justifications. In particular, the already provided results will play a key role in the following reasoning.

***Proof of Theorem 3.*** *Step A (measures depending on a parameter)* – If $\{\mu_s\}_{s\in[0,t]}$ is a family of complex measures in $\mathscr{M}(\mathcal{H}_t)$, then $\max_{s\in[0,t]} \|\mu_s\| < \infty$, since $[0,t]$ is compact and each of the $\mu_s$ has finite total variation. Therefore the functions $s \mapsto \mu_s(B)$ are Lebesgue integrable on $[0,t]$, for all Borel sets $B \in \mathcal{B}(\mathcal{H}_t)$. It is now easy to see, that the set function on $\mathcal{B}(\mathcal{H}_t)$, defined by

$$\Big(\int_0^t \mu_s\, ds\Big)(B) := \int_0^t \mu_s(B) ds \qquad (a.1)$$

defines a complex measure, since by dominated convergence and countable additivity of the $\mu_s$, we have $(\int_0^t \mu_s ds)(\cup_n B_n) = \sum_n \int_0^t \mu_s(B_n) ds$. Moreover, by the usual procedure of extending a result that holds for simple functions to bounded ones, by pointwise approximation, and (in the case of complex measures) using dominated convergence, we find

$$\int_{\mathcal{H}_t} f\ d\Big(\int_0^t \mu_s\, ds\Big) \;=\; \int_0^t \Big(\int_{\mathcal{H}_t} f\, d\mu_s\Big)\, ds \qquad (a.2)$$

for any bounded measurable function $f$ on $\mathcal{H}_t$, which in the case of $f = \mathbb{1}_B$ reduces to the definition of $\int_0^t \mu_s\, ds$.

*Step B* – For fixed $\xi \in \mathbb{R}^d$ and all $s \in [0,t]$, let $\kappa, \mu_s$ and $\nu_s$ be the measures in $\mathscr{M}(\mathcal{H}_t)$ whose Fourier transforms are given by

$$x \mapsto \varphi(x(0)+\xi)\,, \quad x \mapsto V(x(s)+\xi) \quad \text{and} \quad x \mapsto e^{-i\int_s^t V(x(u)+\xi)}\, du \qquad (b.1)$$

respectively. With these choices, it follows from Proposition 1, that (2.28) is the Fourier transform of $\nu_0 * \kappa$. Therefore, denoting the normalized Fresnel integral $\mathcal{F}^{\rho}_{\mathcal{H}_t}$ in the case $\rho = 1$ by $\mathcal{F}_t$, Theorem 2 implies

$$\psi_t(\xi) = \mathcal{F}_t\Big(x \mapsto e^{-i\int_0^t V(x(s)+\xi) ds} \varphi(x(0)+\xi)\Big) \;=\; \int_{\mathcal{H}_t} e^{-\frac{i}{2}\|x\|_t^2}\, d(\nu_0 * \kappa)(x) \qquad (b.2)$$

Since both $V \in \mathscr{F}(\mathbb{R}^d)$ and any $y \in \mathcal{H}_t$ are continuous, the function on $[0,t]$, given by $f(s) := e^{-i\int_s^t V(y(u))\, du}$ is continuously differentiable with

$$f'(s) = iV(y(s))\, e^{-i\int_s^t V(y(u))\, du}$$

for all $s \in [0,t]$. Thus $f(0) = f(1) - \int_0^t f'(s)\, ds$, by the fundamental theorem of calculus. For the shifted path $y(u) = x(u) + \xi$, this gives

$$e^{-i\int_0^t V(x(u)+\xi)\, du} \;=\; 1 - i\int_0^t V(x(s)+\xi)\, e^{-i\int_s^t V(x(u)+\xi) du}\, ds \qquad (b.3)$$

which can also be written in the form $\hat{\nu}_0(x) = \hat{\delta}_0(x) - i\int_0^t \hat{\mu}_s(x)\hat{\nu}_s(x) ds$. As a direct consequence of $(a.2)$, the Fourier transform of $\int_0^t \mu_s ds$, evaluated at $x \in \mathcal{H}_t$, coincides with $\int_0^t \hat{\mu}_s(x) ds$ for any family of complex measures $\{\mu_s\}_{s\in[0,t]}$ on $\mathcal{H}_t$. Thus, by the uniqueness of the Fourier transform and Proposition 1, equation $(b.3)$ gives

$$\nu_0 \;=\; \delta_0 - i\int_0^t \mu_s * \nu_s\, ds \qquad (b.4)$$

In $(b.2)$ we need to integrate with respect to the convolution of $\nu_0$ and $\kappa$, which now can be expressed in terms of $\delta_0, \mu_s$ and $\nu_s$. First, for any Borel set $A \in \mathcal{B}(\mathcal{H}_t)$, we have $\delta_0 * \kappa(A) = \int \mathbb{1}_A(x+y)\, d(\delta_0 \times \kappa)(x,y) = \kappa(A)$ due to Fubini's theorem and the finiteness of complex measures. Moreover, this also allows us to interchange convolution with the parameter integral: For $\mu$, $\{\lambda_s\}_{s\in[0,t]} \subset \mathscr{M}(\mathcal{H}_t)$, we have

$$\Big(\mu * \int_0^t \nu_s\, ds\Big)(A) = \int \mathbb{1}_A(x+y)\, d\Big(\mu \times \int_0^t \nu_s\, ds\Big)(x,y)$$

which by Fubini's theorem can be written as a double integral with respect to $\mu$ and $\int_0^t \nu_s\, ds$ and by using $(a.2)$, it equals $\int_0^t (\mu * \nu_s)(A)\, ds$. Hence, by expression $(b.4)$ and equation $(b.2)$, we obtain

$$\begin{aligned} \psi_t(\xi) &= \int_{\mathcal{H}_t} e^{-\frac{i}{2}\|x\|_t} d\kappa(x) - i \int_{\mathcal{H}_t} e^{-\frac{i}{2}\|x\|_t^2}\, d\Big(\int_0^t \mu_s * \nu_s * \kappa\, ds\Big)(x) \\ &= \mathcal{F}_t\big(x \mapsto \varphi(x(0)+\xi)\big) - i \int_0^t \mathcal{F}_t(f_u^\xi)\, du \end{aligned} \qquad (b.5)$$

where $f_u^\xi(x) := e^{-i\int_u^t V(x(s)+\xi) ds}\, V(x(u)+\xi)\, \varphi(x(0)+\xi)$, and we used $(a.2)$ and the Cameron-Martin type formula (Theorem 2) from the other direction.

*Step C (decomposition in time) – If, for a given $u \in [0,t]$, we cut $[0,t]$ into two parts, $[0,u]$ and $[u,t]$, then the integrand in the second term in $(b.5)$, $\mathcal{F}_t(f_u^\xi)$, equals*

$$\mathcal{F}_{t-u}\Big(x_2 \mapsto e^{-i\int_0^{t-u} V(x_2(s)+\xi)\, ds}\, V(x_2(0)+\xi)\, \mathcal{F}_u\big(x_1 \mapsto \varphi(x_1(0)+x_2(0)+\xi)\big)\Big) \qquad (*)$$

The following proof of this identity relies heavily on the Cameron-Martin type formula (Theorem 2) and the fact that all involved functions belong to a certain Fresnel class. It is based on a method used in the proof of [6, Proposition 2.4]. See [12] for a different proof, which directly uses the definition of normalized Fresnel integrals on $\mathcal{H}_t$ as limits of finite-dimensional oscillatory integrals on the space of piecewise linear functions (Examples 2 and 3).

*Proof of* $(*)$: Consider the linear map $T$ from $\mathcal{H}_t$ to the product space $\mathcal{H}_u \times \mathcal{H}_{t-u}$, given by $Tx := (T_1x, T_2x)$, where $T_1$ and $T_2$ are the linear maps on $\mathcal{H}_t$ cutting $x$ at $u$ into functions from $\mathcal{H}_u$ and $\mathcal{H}_{t-u}$ respectively, i.e.

$$(T_1x)(s) := x(s) - x(u)\,, \quad (T_2x)(s) := x(u+s)$$

Then, obviously $T_1x \in H^1(0,u;\mathbb{R}^d)$, $T_2x \in H^1(0,t-u;\mathbb{R}^d)$ and $(T_1x)(u) = 0$ as well as $(T_2x)(t-u) = 0$, hence $T(\mathcal{H}_t) \subset \mathcal{H}_u \times \mathcal{H}_{t-u}$. Moreover, if we define

$$S : \mathcal{H}_u \times \mathcal{H}_{t-u} \to \mathcal{H}_t\,,\ S(x_1,x_2)(s) = \begin{cases} x_1(s) + x_2(0) & , s \in [0,u] \\ x_2(s-u) & , s \in [u,t] \end{cases}$$

then $ST = \mathbb{1}$ on $\mathcal{H}_t$, $TS = \mathbb{1}$ on $\mathcal{H}_u \times \mathcal{H}_{t-u}$ and therefore $T$ is bijective, with $T^{-1} = S$.

For any $y \in \mathcal{H}_t$, it holds

$$\begin{aligned}
\langle T^{-1}(x_1, x_2), y\rangle_t &= \int_0^u x_1'(s) y'(s)\, ds + \int_u^t x_2'(s-u) y'(s)\, ds \\
&= \int_0^u x'(s)(T_1 y)'(s)\, ds + \int_0^{t-u} x_2'(s)(T_2 y)'(s)\, ds \\
&= \langle x_1, T_1 y\rangle_u + \langle x_2, T_2 y\rangle_{t-u}
\end{aligned} \tag{c.1}$$

and therefore, the inner product on $\mathcal{H}_u \times \mathcal{H}_{t-u}$ given by $\langle x_1, y_1\rangle_u + \langle x_2, y_2\rangle_{t-u}$, satisfies

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \langle T^{-1}(x_1, x_2), T^{-1}(y_1, y_2)\rangle_t \tag{c.2}$$

If, for any $s \in [0, t]$, the measures $\kappa, \mu_s$ and $\nu_s$ are those defined in step 1, then we have $f_u^\xi = \hat{\alpha}_u\, \hat{\kappa}$, where $\alpha_u := \nu_u * \mu_u$ by Proposition 1. Due to the Cameron-Martin type formula (Theorem 2), it holds

$$\mathcal{F}_t(f_u^\xi) = \int_{\mathcal{H}_t} e^{-\frac{i}{2}\|x\|_t^2}\, d(\alpha_u * \kappa)(x)$$

Therefore, by using equation (2.23) and ($c$.2), we find

$$\begin{aligned}
\mathcal{F}_t(f_u^\xi) &= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} e^{-\frac{i}{2}\|T^{-1}(x_1,x_2)\|_t^2}\, d\big((\alpha_u * \kappa)\circ T^{-1}\big)(x_1, x_2) \\
&= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} e^{-\frac{i}{2}\|x_1\|_u^2}\, e^{-\frac{i}{2}\|x_2\|_{t-u}^2}\, d\big((\alpha_u * \kappa)\circ T^{-1}\big)(x_1, x_2)
\end{aligned} \tag{c.3}$$

By performing a shift in the $ds$-integral inside of the first term in the expression for $f_u^\xi(x)$, we find $\hat{\alpha}_u(T^{-1}(x_1, x_2)) = e^{-i\int_0^{t-u} V(x_2(s)+\xi)\, ds}\, V(x_2(0) + \xi)$. Hence, by Proposition 4 there is a measure $\lambda_u \in \mathscr{M}(\mathcal{H}_{t-u})$, such that

$$\hat{\alpha}_u(T^{-1}(x_1, x_2)) = \hat{\lambda}_u(x_2) \tag{c.4}$$

for all $x_1 \in \mathcal{H}_u$ and $x_2 \in \mathcal{H}_{t-u}$. Therefore, by Proposition 1

$$\hat{\lambda}_u(x_2)\, \hat{\kappa}(T^{-1}(x_1, x_2)) = \widehat{\alpha_u * \kappa}(T^{-1}(x_1, x_2))$$

Since for the Dirac measure $\delta_0$ on $\mathcal{H}_u$, the Fourier transform of $\delta_0 \times \lambda_u$, evaluated at $(x_1, x_2) \in \mathcal{H}_u \times \mathcal{H}_{t-u}$, just equals $\hat{\lambda}_u(x_2)$, and due to the uniqueness of the Fourier transform and Proposition 1, this shows

$$(\delta_0 \times \lambda_u) * (\kappa \circ T^{-1}) = (\alpha_u * \kappa)\circ T^{-1} \tag{c.5}$$

where we also used the fact that for any $\mu \in \mathscr{M}(\mathcal{H}_t)$, by ($c$.2) it holds

$$\begin{aligned} \widehat{\mu \circ T^{-1}}(x_1, x_2) &= \int_{\mathcal{H}_t} e^{i\langle (x_1,x_2), Ty\rangle} \, d\mu(y) \\ &= \int_{\mathcal{H}_t} e^{i\langle T^{-1}(x_1,x_2), y\rangle_t} \, d\mu(y) \;=\; \hat{\mu} \circ T^{-1}(x_1, x_2) \end{aligned} \tag{$c$.6}$$

Before we continue with plugging in ($c$.5) in equation ($c$.3), we need to introduce complex measures $\{\kappa_1^{x_2}\}_{x_2 \in \mathcal{H}_{t-u}}$ and $\kappa_2$, which will be usefull in the further calculation. If, for all Borel sets $A \subset \mathcal{H}_u$ and $B \subset \mathcal{H}_{t-u}$, we set

$$\begin{aligned} \kappa_1^{x_2}(A) &:= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} \mathbb{1}_A(y_1) \, e^{i\langle x_2, y_2\rangle_{t-u}} \, d(\kappa \circ T^{-1})(y_1, y_2) \\ \kappa_2(B) &:= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} e^{-\frac{i}{2}\|y_1\|_u^2} \, \mathbb{1}_B(y_2) \, d(\kappa \circ T^{-1})(y_1, y_2) \end{aligned}$$

then $\kappa_1^{x_2} \in \mathscr{M}(\mathcal{H}_u)$ for all $x_2 \in \mathcal{H}_{t-u}$ and $\kappa_2 \in \mathscr{M}(\mathcal{H}_{t-u})$, where countable additivity follows from dominated convergence and the finite total variation. It is an easy application of the method, which we now have used several times, of carrying a result for simple functions over to bounded Borel measurable ones by pointwise approximation, to get the identities

$$\begin{aligned} \int_{\mathcal{H}_u} h(y_1) \, d\kappa_1^{x_2}(y_1) &= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} h(y_1) \, e^{i\langle x_2, y_2\rangle_{t-u}} \, d(\kappa \circ T^{-1})(y_1, y_2) \\ \int_{\mathcal{H}_{t-u}} g(y_2) \, d\kappa_2(y_2) &= \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} g(y_2) \, e^{-\frac{i}{2}\|y_1\|_u^2} \, d(\kappa \circ T^{-1})(y_1, y_2) \end{aligned} \tag{$c$.7}$$

for all bounded Borel functions $h$ and $g$ on $\mathcal{H}_u$ and $\mathcal{H}_{t-u}$ respectively. After using ($c$.5) in ($c$.3), writing the convolution in terms of an integral on the product space $(\mathcal{H}_u \times \mathcal{H}_{t-u}) \times (\mathcal{H}_u \times \mathcal{H}_{t-u})$ and applying Fubini's theorem, we get

$$\mathcal{F}_t(f_u^\xi) \;=\; \int_{\mathcal{H}_u \times \mathcal{H}_{t-u}} \int_{\mathcal{H}_{t-u}} e^{-\frac{i}{2}\|y_1\|_u^2} \, e^{-\frac{i}{2}\|x_2 + y_2\|_{t-u}^2} d\lambda_u(x_2) \, d(\kappa \circ T^{-1})(y_1, y_2)$$

From another application of Fubini's theorem and equation ($c$.7), it follows

$$\begin{aligned} \mathcal{F}_t(f_u^\xi) &= \int_{\mathcal{H}_{t-u}} \int_{\mathcal{H}_{t-u}} e^{-\frac{i}{2}\|x_2 + y_2\|_{t-u}^2} \, d\kappa_2(y_2) \, d\lambda_u(x_2) \\ &= \int_{\mathcal{H}_{t-u}} e^{-\frac{i}{2}\|x_2\|_{t-u}^2} \, d(\kappa_2 * \lambda)(x_2) \;=\; \mathcal{F}_{t-u}(\hat{\kappa}_2 \, \hat{\lambda}_u) \end{aligned} \tag{$c$.8}$$

where the last equality follows from Proposition 1 and the Cameron-Martin type formula. The identities in ($c$.7) give

$$\begin{aligned}\hat{\kappa}_2(x_2) &= \int_{\mathcal{H}_{t-u}} e^{i\langle x_2,y_2\rangle_{t-u}}\, d\kappa_2(y_2) \\ &= \int_{\mathcal{H}_u\times\mathcal{H}_{t-u}} e^{i\langle x_2,y_2\rangle_{t-u}}\, e^{-\frac{i}{2}\|y_1\|_u^2}\, d(\kappa\circ T^{-1})(y_1,y_2) \\ &= \int_{\mathcal{H}_u} e^{-\frac{i}{2}\|y_1\|_u^2}\, d\kappa_1^{x_2}(y_1) \;=\; \mathcal{F}_u\Big(\widehat{\kappa_1^{x_2}}\Big)\end{aligned} \qquad (c.9)$$

By ($c$.6) and the definition of $\kappa$,

$$\begin{aligned}\widehat{\kappa_1^{x_2}}(x_1) &= \int_{\mathcal{H}_u} e^{i\langle x_1,y_1\rangle_u}\, d\kappa_1^{x_2}(x_1) \\ &= \int_{\mathcal{H}_u\times\mathcal{H}_{t-u}} e^{i(\langle x_1,y_1\rangle_u+\langle x_2,y_2\rangle_{t-u})}\, d(\kappa\circ T^{-1})(y_1,y_2) \\ &= \hat{\kappa}(T^{-1}(x_1,x_2)) \;=\; \varphi(T^{-1}(x_1,x_2)(0)+\xi)\end{aligned} \qquad (c.10)$$

Putting together equations ($c$.8), ($c$.9) and ($c$.10) gives

$$\mathcal{F}_t(f_u^\xi) \;=\; \mathcal{F}_{t-u}\Big(x_2\mapsto \hat{\lambda}_u(x_2)\,\mathcal{F}_u\big(x_1\mapsto\varphi(x_1(0)+x_2(0)+\xi)\big)\Big)$$

which, by ($c$.4), coincides exactly with equation ($*$). Let us remark, that equation (c.9) shows that the function $x_2\mapsto\mathcal{F}_u\big(x_1\mapsto\varphi(x_1(0)+x_2(0)+\xi)\big)$ belongs to $\mathscr{F}(\mathcal{H}_{t-u})$ and from this result, it is an easy task to show that the function $\zeta\mapsto\mathcal{F}_u(x_1\mapsto\varphi(x_1(0)+\zeta))$ is in $\mathscr{F}(\mathbb{R}^d)$. Thus, for each $u>0$, the bounded linear map $U_0(u)$—which was introduced in Proposition 5 and shown to be equal to $e^{-iuH_0}$ in Proposition 6—actually maps $\mathscr{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ into $\mathscr{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.

*Step D (strong solution)* – We can finally use Propositions 5-9, the decomposition ($b$.5) and equation ($*$) to show that (2.29) forms a strong solution of the Schrödinger equation (2.30). For any $t>0$, let $U(t)$ be the *bounded* linear map (see Proposition 7) on $\mathscr{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, defined in Proposition 7 by the FFPI

$$\big(U(t)\varphi\big)(\xi) \;=\; \mathcal{F}_t\Big(x\mapsto e^{-i\int_0^t V(x(s)+\xi)ds}\,\varphi(x(0)+\xi)\Big)$$

for almost every $\xi\in\mathbb{R}^d$. By Propositions 5 and 6, it holds for all $\xi\in\mathbb{R}^d$ and $u>0$,

$$\mathcal{F}_u\big(x\mapsto\varphi(x(0)+\xi)\big) \;=\; \big(U_0(u)\varphi\big)(\xi) \;=\; \big(e^{-iuH_0}\varphi\big)(\xi)$$

This, together with the fact that $Ve^{-iuH_0}\varphi$ belongs to $\mathscr{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ (remark at the end of Step C for being in the Fresnel class $\mathscr{F}(\mathbb{R}^d)$, and boundedness of $V$ for square integrability), allows to write equation ($*$) in the more compact form

$$\mathcal{F}_t(f_u^\xi) = \big(U(t-u)Ve^{-iuH_0}\varphi\big)(\xi)$$

Hence, by using ($b$.5) and Proposition 8, we arrive at the integral equation

$$\begin{aligned} U(t)\varphi &= e^{-itH_0}\varphi - i\int_0^t U(t-u)Ve^{-iuH_0}\varphi\,du \\ &= e^{-itH_0}\varphi - i\int_0^t U(s)Ve^{-i(t-s)H_0}\varphi\,ds \end{aligned} \qquad (d.1)$$

where the second equality is due to the simple change of variables, $s = t - u$. After performing $N$ iterations, we obtain

$$\sum_{k=0}^{N}(-i)^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} e^{-it_kH_0}Ve^{-i(t_{k-1}-t_k)H_0}V\cdots e^{-i(t-t_1)H_0}\varphi\,dt_k\cdots dt_1 + I_N$$

for $U(t)\varphi$, where $k = 0$ labels the free propagator term of equation ($d$.1), and

$$I_N := \int_0^t \int_0^{t_1} \cdots \int_0^{t_N} U(t_{N+1})Ve^{-i(t_N-t_{N+1})H_0}V\cdots e^{-i(t-t_1)H_0}\varphi\,dt_{N+1}\cdots dt_1$$

Due to $\|U(t_{N+1})\| \leqslant e^{t\|\mu\|}$ whenever $t_{N+1} \leqslant t$ (see Proposition 7), where $\mu \in \mathcal{M}(\mathbb{R}^d)$ s.th. $\hat{\mu} = V$, it follows

$$\|I_N\|_2 \leqslant e^{t\|\mu\|}\frac{t^{N+1}\|V\|_\infty^{N+1}}{(N+1)!}\|\varphi\|_2 \xrightarrow{N\to\infty} 0$$

Therefore, from ($d$.1), by setting $V(s) := e^{isH_0}Ve^{-isH_0}$, it follows

$$\left\| U(t)\varphi - \sum_{k=0}^{N}(-i)^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} e^{-itH_0}V(t-t_k)\cdots V(t-t_1)\varphi\,dt_k\cdots dt_1 \right\|_2 \xrightarrow{N\to\infty} 0$$

If we perform in each term of the sum $k$ changes of variables, the new set of variables being given by $s_j := t - t_{k+1-j}$, $j = 1, \ldots, k$, then integrating $t_j$ in $[0, t_{j-1}]$ is replaced by $s_l$ being integrated in $[t - t_{j-1}, t] = [s_{l+1}, t]$, where $l = k + 1 - j$. Hence

$$U(t)\varphi = \sum_{k=0}^{\infty}(-i)^k \int_0^t \int_{s_k}^t \cdots \int_{s_2}^t e^{-itH_0}V(s_1)\cdots V(s_k)\varphi\,ds_1\cdots ds_k$$

The $L^2$-continuity of $U_0(t) = e^{-itH_0}$ allows it to be interchanged with each $L^2$-valued Riemann integral as well as being pulled out of the series. Thus, we have found

$$U(t)\varphi = U_0(t)A(t)\varphi \qquad (d.2)$$

for all $\varphi \in \mathscr{F}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, where $A(t) := A(t,0)$, and $(A(t,s))_{s,t\geqslant 0}$ is the unitary propagator on $L^2(\mathbb{R}^d)$ introduced in Proposition 9, which satisfies $\frac{d}{dt}A(t)\varphi = -i\,V(t)A(t)\varphi$. Now, under the additional assumption $\varphi \in H^2(\mathbb{R}^d)$, this shows that $U(t)\varphi \in H^2(\mathbb{R}^d)$, because from Lemma 6, we know that $A(t)\varphi \in H^2(\mathbb{R}^d)$ and $U_0(t) = e^{-itH_0}$ always leaves the domain of $H_0$ invariant.

In order to find the strong derivative of $U(t)\varphi$, let $\varDelta_h B(t) := B(t+h)-B(t)$, for any $t$-dependent linear map $B(t)$. Then, we need to show, that $\frac{1}{h}\varDelta_h(U_0(t)A(t))\varphi$ converges in $L^2(\mathbb{R}^d)$ as $h \to 0$, because in this case, by definition,

$$\Big\| \frac{d}{dt}U(t)\varphi - \frac{1}{h}\varDelta_h(U_0(t)A(t))\varphi \Big\|_2 \xrightarrow{h\to 0} 0 \qquad (d.3)$$

We have $\varDelta_h(U_0(t)A(t)) = (\varDelta_h U_0)(t)A(t) + U_0(t)(\varDelta_h A)(t) + (\varDelta_h U_0)(t)(\varDelta_h A)(t)$, and for any $f \in L^2(\mathbb{R}^d)$,

$$\Big\| \frac{1}{h}\varDelta_h U_0 \varDelta_h A f \Big\|_2^2 \leqslant 2 \Big\| \frac{\varDelta_h U_0}{h}\frac{\varDelta_h A}{h} f \Big\|_2 \|\varDelta_h A f\|_2 \qquad (d.4)$$

where, in favor of notation, we did omit the $t$-dependencies. Now, let $\alpha_h(t)$ be given by $\alpha_h := \frac{1}{h}\varDelta_h U_0$, then $\|\alpha_h g - \frac{d}{dt}U_0 g\|_2 \to 0$, as $h \to 0$, for all $g \in \mathcal{D}(H_0) = H^2(\mathbb{R}^d)$. In particular, $\|\alpha_h g\|_2$ converges as $h \to 0$, and therefore is locally uniformly bounded in $h$, i.e. there is a compact intervall $I \subset \mathbb{R}$ containing 0, and a constant $C \geqslant 0$, s.th. $\sup_{h\in I}\|\alpha_h g\|_2 \leqslant C$. Denoting $\frac{1}{h}\varDelta_h A$ by $\beta_h$, since by Lemma 6, $A(t)$ leaves $H^2(\mathbb{R}^d)$ invariant whenever $V \in C^2(\mathbb{R}^d)$ with bounded derivatives, it follows that for each $h' \in I$, there is a constant $C_{h'}$, with $\sup_{h\in I}\|\alpha_h \beta_{h'} f\|_2 \leqslant C_{h'}$, and thus, for all $h \in I$,

$$\Big\| \frac{\varDelta_h U_0}{h}\frac{\varDelta_h A}{h} f \Big\|_2 \leqslant \sup_{h\in I}\|\alpha_h \beta_h f\|_2 \leqslant \sup_{h'\in I}\sup_{h\in I}\|\alpha_h \beta_{h'} f\|_2 \leqslant \sup_{h'\in I} C_{h'} =: C < \infty$$

due to the compactness of $I$. Hence, from the strong continuity of $(A(t))_{t\geqslant 0}$ and $(d.4)$, it follows

$$\Big\| \frac{1}{h}\varDelta_h U_0 \varDelta_h A f \Big\|_2^2 \leqslant 2C\,\|A(t+h)f - A(t)f\|_2 \xrightarrow{h\to 0} 0$$

Thus, only the first two terms in the above expression for $\varDelta_h(U_0(t)A(t))$ contribute to the limit $(d.3)$. From $\frac{d}{dt}U_0(t)f = -iH_0 U_0(t)f$ as well as $\frac{d}{dt}A(t)f = -iV(t)A(t)f$ for all $f \in H^2(\mathbb{R}^d)$, from Lemma 6 and from the boundedness of $U_0(t)$, it follows

$$\lim_{h\to 0}\frac{1}{h}\varDelta_h(U_0(t)A(t))\varphi = -iH_0U_0(t)A(t)\varphi - iU_0(t)V(t)A(t)\varphi = -iH\,U(t)\varphi$$

where $H = H_0 + V$ with $\mathcal{D}(H) = H^2(\mathbb{R}^d)$ and $U_0(t)V(t) = VU_0(t)$, which follows from $U_0(t) = e^{-itH_0}$ and the definition of $V(t)$. Together with $(d.4)$, this shows that $U(t)\varphi$ is a strong solution to the Schrödinger equation (2.30).

# 3 The method of stationary phase

Usually, the *method of stationary phase* or *stationary phase approximation* refers to the study of the asymptotic behaviour of oscillatory integrals with phase functions depending on a parameter $h \in \mathbb{R}$, that controls the stength of oscillation of the integrand.

In fact, the total cancellation in several regions in the integration domain due to strong oscillations of the phase factor, is one of the core features of oscillatory integrals. Making the phase function parameter dependent allows to get control over these oscillations, which then helps to extract the origin of the main contributions to the total integral.

For example, consider a given oscillatory integral of the form $\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x)} f(x)dx$, with $\phi$ real-valued, and consider the case when $h$ approaches zero. In regions, where $\phi$ varies a lot, the oscillations will be very strong, as will be the cancellations. Whereas, no or only slight variations of $\phi$ result in less oscillations, and therefore those regions will have the most effect on the total value of the integral. As we will see below, the leading contribution comes from the so called *critical set*

$$\mathcal{C}_\phi \;:=\; \{x \in \mathbb{R}^d \mid \nabla\phi(x) = 0\}$$

## 3.1 Stationary phase approximation of Fresnel integrals

In the following, if $u : \mathcal{H} \to \mathbb{C}$ is a function on a real separable Hilbert space $\mathcal{H}$, then we will denote its *Fréchet derivative* by $Du$, whenever it exists, i.e. if for each $x \in \mathcal{H}$ there is a *bounded linear map* $L_x : \mathcal{H} \to \mathbb{C}$ such that

$$\frac{1}{\|h\|}\left|u(x+h) - u(x) - L_x(h)\right| \xrightarrow{\|h\|\to 0} 0 \tag{3.1}$$

If the limit exists, then $L_x$ is *unique*[6], and $Du(x) := L_x$.

As simple consequences of this definition, we obtain Propositions 10-12, whose results have been used in [2] without going into the details of the calculation.

**Proposition 10**. *If $\mu \in \mathcal{M}(\mathcal{H})$ satisfies $\int \|\alpha\|\, d|\mu|(\alpha) < \infty$, then the function $V := \hat{\mu}$ is Fréchet differentiable, with $DV(x) = i\int \langle\alpha, \cdot\rangle e^{i\langle x,\alpha\rangle} d\mu(\alpha)$.*

[6] If $A_x, B_x$ are bounded linear maps from $\mathcal{H}$ to $\mathbb{C}$ satisfying (3.1), then for a given $\varepsilon > 0$, there exists $\delta > 0$, such that $|\Delta_h u(x) - A_x h| \leqslant \varepsilon\|h\|/2$ as well as $|\Delta_h u(x) - B_x h| \leqslant \varepsilon\|h\|/2$, whenever $\|h\| < \delta$ and $\Delta_h u(x) := u(x+h) - u(x)$. Hence $|(A-B)(h)| \leqslant |A(h) - \Delta_h u(x)| + |\Delta_h u(x) - B(h)| < \varepsilon\|h\|$. So, dividing by $\|h\|$ shows $\|A_x - B_x\| \leqslant \varepsilon$, where $\varepsilon$ was arbitrary. Therefore $A_x = B_x$.

***Proof.*** Boundedness follows immediately from Cauchy-Schwarz's inequality. Moreover, it holds $|V(x+h)-V(x)-i\int\langle\alpha,h\rangle e^{i\langle x,\alpha\rangle}d\mu(\alpha)| \leqslant \int|e^{i\langle h,\alpha\rangle}-1-i\langle\alpha,h\rangle|\,d|\mu|(\alpha)$. Taylor's theorem implies $|e^{it}-1-it| \leqslant r(t)\,|t|$ for all $t\in\mathbb{R}$, where $r$ is a bounded function on $\mathbb{R}$ with $r(t)\to 0$ as $t\to 0$. Thus, $\int|e^{i\langle h,\alpha\rangle}-1-i\langle\alpha,h\rangle|\,d|\mu|(\alpha) \leqslant \int\|\alpha\|r(\langle h,\alpha\rangle)d|\mu|(\alpha)\,\|h\|$. Due to the boundedness of $r$ and $\int\|\alpha\|d|\mu|(\alpha)<\infty$, the claim follows from the theorem of dominated convergence and $r(\langle h,\alpha\rangle)\to 0$ as $\|h\|\to 0$. □

We will also use the *second Fréchet derivative*, which by definition is the Fréchet derivative of $Du:\mathcal{H}\to\mathcal{L}(\mathcal{H},\mathbb{C})$, where $\mathcal{L}(V,U)$ denotes the space of bounded linear maps between normed spaces $V$ and $U$. Thus, $D^2u$ is defined as the bounded linear map from $\mathcal{H}$ to $\mathcal{L}\big(\mathcal{H},\mathcal{L}(\mathcal{H},\mathbb{C})\big)$ satisfying

$$\frac{1}{\|h\|}\left\|Du(x+h)-Du(x)-(D^2u(x))(h)\right\|_{\mathcal{L}(\mathcal{H},\mathbb{C})} \xrightarrow{\|h\|\to\infty} 0 \tag{3.2}$$

**Proposition 11**. *If $\mu\in\mathcal{M}(\mathcal{H})$ satisfies $\int\|\alpha\|^k\,d|\mu|(\alpha)<\infty$ for $k=1,2$, then $V{:=}\hat{\mu}$ is twice Fréchet differentiable, and it holds*

$$(D^2V(x))(y) \;=\; -\int_{\mathcal{H}}\langle\alpha,\cdot\rangle\langle\alpha,y\rangle e^{i\langle x,\alpha\rangle}d\mu(\alpha) \tag{3.3}$$

***Proof.*** Let $r$ be chosen as in the proof of Proposition 10, then we have $\big|(Df(x+h))(y)-(Df(x))(y)+\int\langle\alpha,h\rangle\langle\alpha,y\rangle e^{i\langle x,\alpha\rangle}d\mu(\alpha)\big| \leqslant \int|\langle\alpha,y\rangle|\,|e^{i\langle h,\alpha\rangle}-1-i\langle\alpha,h\rangle|\,d|\mu|(\alpha)$. It then follows from the Cauchy-Schwarz inequality, that

$$\|h\|^{-1}\left\|DV(x{+}h)-DV(x)+\int\langle\alpha,\cdot\rangle\langle\alpha,y\rangle e^{i\langle x,\alpha\rangle}d\mu(\alpha)\right\|_{\mathcal{L}(\mathcal{H},\mathbb{C})} \leqslant \int\|\alpha\|^2 r(\langle\alpha,h\rangle)\,d|\mu|(\alpha)$$

The boundedness of $r$ together with $\int\|\alpha\|^2d|\mu|(\alpha)<\infty$, allows to apply the theorem of dominated convergence, which then proves the claim, since $r(\langle\alpha,h\rangle)\to 0$ as $\|h\|\to 0$.□

**Proposition 12** *If $V=\hat{\mu}\in\mathscr{F}(\mathcal{H})$ is such that $\int_{\mathcal{H}}\|\alpha\|^2d|\mu|(\alpha)=:C<\infty$, then for any $a\in\mathcal{H}$, $D^2V(a)$, considered as an operator acting on $\mathcal{H}$ (by using Riesz representation theorem), is of trace class, with trace norm not exceeding $C$.*

***Proof.*** From Proposition 11 and Riesz representation theorem, $D^2V(a)$ may be considered to be an operator on $\mathcal{H}$ such that $\langle x,D^2V(a)y\rangle=-\int_{\mathcal{H}}\langle x,\alpha\rangle\langle\alpha,y\rangle\mu^a(\alpha)$ for all $x,y\in\mathcal{H}$. By [37, 3.22], its trace norm is

$$\|D^2V(a)\|_1 \;=\; \sup\left\{\sum_{n\in\mathbb{N}}\big|\langle e_n,D^2V(a)f_n\rangle\big| : \{e_n\}_{n\in\mathbb{N}},\{f_n\}_{n\in\mathbb{N}}\text{ ONB in }\mathcal{H}\right\} \tag{3.4}$$

and, by monotone convergence,

$$\sum_{n\in\mathbb{N}}\big|\langle e_n,D^2V(a)f_n\rangle\big| \leqslant \int_{\mathcal{H}}\sum_n|\langle e_n,\alpha\rangle\langle\alpha,f_n\rangle|\,d|\mu|(\alpha)$$

From the Cauchy-Schwarz inequality on $l^2$, and Parseval's identity, it follows

$$\|D^2V(a)\|_1 \leqslant \int_{\mathcal{H}} \Big(\sum_{n\in\mathbb{N}} |\langle e_n,\alpha\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\in\mathbb{N}} |\langle f_n,\alpha\rangle|^2\Big)^{\frac{1}{2}} d|\mu|(\alpha) \;=\; \int_{\mathcal{H}} \|\alpha\|^2 d|\mu|(\alpha)$$

which shows that $D^2V(a)$ is trace class and $\|D^2V(a)\|_1 \leqslant C$. □

For the statement of Theorem 4 below, we need to introduce two more or less unconnected notions, *stationary points* of a Fréchet differentiable function on $\mathcal{H}$, and the *Fredholm determinant* of $\mathbb{1}+A$, where $A$ is a trace class operator on $\mathcal{H}$.

**Definition 6** *(Stationary point).* If $a \in \mathcal{H}$ satisfies $Du(a) = 0$, for a given Fréchet differentiable function $u : \mathcal{H} \to \mathbb{C}$, then $a$ is called a *stationary point* of $u$.

In the case $\mathcal{H} = \mathbb{R}^d$, the Fréchet derivative of a differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ at $x \in \mathbb{R}^d$, is the linear map given by $Du(x) = \langle \nabla u(x), \cdot\,\rangle_{\mathbb{R}^d}$. Therefore, in this case, stationary points of $u$ are solutions to the equation $\nabla u(a) = 0$.

Next, for a trace class operator $A$ on $\mathcal{H}$, we use the following definition of the Fredholm determinant of $\mathbb{1}+A$, which is due to [14]. It relies on the density of finite-rank operators in the trace class, with respect to the trace norm, and [14, Lemma 3.1], which says: *If $(A_n)_{n=1}^{\infty}$ is a sequence of finite-rank operators on $\mathcal{H}$ converging to $A$ in trace norm, then the sequence $(\det(\mathbb{1}+A_n))_{n=1}^{\infty}$ converges, and its limit is independent of the particular choice of the sequence $(A_n)_{n=1}^{\infty}$.*

**Definition 7** *(Fredholm determinant).* For a given trace class operator $A$ on $\mathcal{H}$, let $(A_n)_{n=1}^{\infty}$ be a sequence of finite-rank operators on $\mathcal{H}$ converging to $A$ in trace norm, then we define the *Fredholm determinant* of $\mathbb{1}+A$ by $\mathrm{Det}(\mathbb{1}+A) := \lim_{n\to\infty} \det(\mathbb{1}+A_n)$.

We have now the ingredients in order to state the key result for the semiclassical approximation of the FFPI in the following section. For the proof, we will give several intermediate statements, which rely on methods used in [4], [5], [1] and [29].

**Theorem 4**. *Let $\mathcal{H}$ be a real separable Hilbert space and $\mu, \nu \in \mathcal{M}(\mathcal{H})$ such that there are constants $C_\mu, C_\nu, \varepsilon > 0$ with*

$$\int_{\mathcal{H}} \|x\|^j d|\mu|(x) \;\leqslant\; C_\mu \frac{j!}{\varepsilon^j} \quad , \quad \int_{\mathcal{H}} \|y\|^j d|\nu|(x) \;\leqslant\; C_\nu \frac{j!}{\varepsilon^j} \tag{3.5}$$

*as well as $12\, C_\mu < \varepsilon^2$. Moreover, let $V, g \in \mathscr{F}(\mathcal{H})$ be given by $V{=}\hat{\mu}$ and $g{=}\hat{\nu}$, then the function $\phi := \frac{1}{2}\|\cdot\|^2 - V$ has a unique stationary point $a \in \mathcal{H}$ and the Fresnel integral $I(h) := \mathcal{F}^h_{\mathcal{H}}\big(x \mapsto e^{-\frac{i}{h}V(x)} g(x)\big)$, for $|h|{<}1$ takes the form*

$$I(h) \;=\; \mathrm{Det}\big(\mathbb{1}{-}D^2V(a)\big)^{-\frac{1}{2}}\, e^{\frac{i}{h}\phi(a)}\, g(a) + R(h) \tag{3.6}$$

*where $|R(h)| \to 0$ as $h \to 0$.*

Let us add the remark, that we restrict ourselves to the case of a unique stationary point, by imposing the condition $12\,C_\mu < \varepsilon^2$. It is however possible to extend the theory to the non-unique case, where the right-hand side of (3.6) is replaced by a sum over the set of stationary points (see [2], [1] and [29]). The uniqueness condition is given by the following Lemma, which is taken from [4, Lemma 2.1].

**Lemma 7**. *If $V{=}\hat{\mu} \in \mathscr{F}(\mathcal{H})$, $\int \|\alpha\|^2 d|\mu|(\alpha) < R$ for some $R < 1$, then the equation $DV(a) = \langle a, \cdot\rangle$ has a unique solution $a \in \mathcal{H}$.*

***Proof.*** For any $x, y, z \in \mathcal{H}$, by Proposition 10,

$$\big\|(DV(x){-}DV(y))(z)\big\| \;\leqslant\; \int_{\mathcal{H}} |\langle \alpha, z\rangle|\, \big|1 - e^{i\langle x-y,\alpha\rangle}\big|\, d|\mu|(\alpha)$$

Using $|1{-}e^{it}| = 2|\sin(x/2)| \leqslant |x|$, we obtain $\|DV(x){-}DV(y)\| \leqslant \int \|\alpha\|^2 d|\mu|(\alpha)\, \|x{-}y\|$. Together with the assumption $\int \|\alpha\|^2 d|\mu|(\alpha) < R$ and $R < 1$, this shows that $DV$ is a contraction mapping from $\mathcal{H}$ to $\mathcal{H}^* = \mathcal{H}$. Thus, by Banach's fixed point theorem, there is a unique fixed point $a \in \mathcal{H}$, i.e. $DV(a) = \langle a, \cdot\rangle$. □

At some point in the proof of Theorem 4, the following simple result from [1] will allow us to reduce the problem to finitely many dimensions, by using the definition of the Fresnel integral in terms of finite-dimensional approximations.

**Lemma 8**. *Let $(J_n)_{n\in\mathbb{N}}$ be a sequence of functions on $\mathbb{R}$, let $J$ be defined on $\mathbb{R}\backslash\{0\}$, such that $|J_n(h){-}J(h)| \to 0$ as $n \to \infty$, for all $h \in \mathbb{R}\backslash\{0\}$, and moreover, assume that for each $n \in \mathbb{N}$, there is a constant $q_n \in \mathbb{C}$ satisfying $|J_n(h) - q_n| \leqslant C\,|h|$ for all $h \in \mathbb{R}$ and some $C > 0$. Then $(J_n(0))_{n\in\mathbb{N}}$ is convergent, and moreover*

$$|J(h) - J(0)| \leqslant C\,|h| \tag{3.7}$$

*for all $h \in \mathbb{R}$, where $J(0) := \lim_{n\to\infty} J_n(0)$.*

***Proof.*** By assumption, $|J_n(0){-}q_n| \leqslant 0$, i.e. $J_n(0) = q_n$. For $\varepsilon > 0$, choose $h_0 \in \mathbb{R}$ with $|h_0| < \varepsilon/(3C)$, then choose $N \in \mathbb{N}$ big enough, such that $|J_n(h_0){-}J_m(h_0)| < \varepsilon/3$ for all $n, m \geqslant N$. Hence

$$|J_n(0){-}J_m(0)| \leqslant |J_n(0){-}J_n(h_0)| + |J_n(h_0){-}J_m(h_0)| + |J_m(h_0){-}J_m(0)| \leqslant 2C|h_0|{+}\frac{\varepsilon}{3} < \varepsilon$$

for all $n, m \geqslant N$. So $(J_n(0))_{n\in\mathbb{N}}$ forms a Cauchy sequence and therefore has a unique limit $J(0)$. Inequality (3.7) then follows from $|J_n(h){-}J_n(0)| \leqslant C|h|$. □

In the proof of Theorem 4, we will need to calculate finite-dimensional oscillatory integrals of more general amplitudes than can be provided by the Fresnel class. This is justified by the following Lemma. Its proof is a slightly corrected version of the one presented in [1, Theorem 3.1].

**Lemma 9**. *Let $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with bounded first order derivatives, $\|\nabla V\|_\infty{<}\infty$, and assume all higher derivatives to be of at most linear growth, i.e. for all $\alpha \in \mathbb{N}_0^d$, there is a constant $m_\alpha{>}0$ such that $|\partial^\alpha V(x)| \leqslant m_\alpha(1+|x|)$ for all $x \in \mathbb{R}^d$. Let $g \in C^\infty(\mathbb{R}^d, \mathbb{C})$ be such that there exists $p \in \mathbb{N}_0$ and $C_\alpha{>}0$ with $|\partial^\alpha g(x)| \leqslant C_\alpha(1+|x|^2)^{\frac{p}{2}}$ for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$. Then*

$$I(h) := \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}\|x\|^2} e^{-\frac{i}{h}V(x)} g(x)\, dx \tag{3.8}$$

*exists, for all $h \in \mathbb{R}$.*

***Proof.*** Let $(\chi_j)_{j=0,1}$ be a partition of unity in $\mathbb{R}^d$, i.e. $\chi_j \in C^\infty(\mathbb{R}^d)$, $0 \leqslant \chi_j \leqslant 1$ for $j{=}0,1$ and $\chi_0{+}\chi_1 = 1$, such that $\operatorname{supp}(\chi_1) \subset B(0, \frac{3}{2}r)$ and $\chi_1|_{B(0,r)} \equiv 1$, where $r$ will be fixed later. For $\psi \in \mathcal{S}^*(\mathbb{R}^d)$, $\psi_\varepsilon := \psi(\varepsilon\,\cdot)$, consider

$$I(h,\psi_\varepsilon) := \int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} e^{-\frac{i}{h}V(x)} g(x)\,\psi(\varepsilon x)\, dx \;=\; I_0(h,\psi_\varepsilon) + I_1(h,\psi_\varepsilon)$$

where $I_j(h,\psi_\varepsilon) := I(h,\psi_\varepsilon\chi_j)$ for $j{=}0,1$. Since $g\chi_1 \in L^1(\mathbb{R}^d)$, from dominated convergence, it follows $I_1(h,\psi_\varepsilon) \to \int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} g(x)\chi_1(x)\, dx$ as $\varepsilon \to 0$. For $I_0(h,\psi_\varepsilon)$, more work is needed. First, define $a : \mathbb{R}^d \to \mathbb{R}^d$ by

$$a_j(x) \;:=\; \frac{\partial_j\varphi(x)}{|\nabla\varphi(x)|^2}\,\chi_0(x)$$

where $\varphi := \frac{1}{2}\|\cdot\|^2 - V$. We will show, that *for any $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$, such that*

$$\big|\partial^\alpha a_j(x)\big| \;\leqslant\; \frac{C_\alpha}{1+|x|} \tag{$*$}$$

*for all $x \in \mathbb{R}^d$.* By the assumptions on $V$, for any $\beta, \gamma \in \mathbb{N}^d$ and $j = 1, \ldots, j$, we have $|\partial^\beta \partial_j\varphi| = |\partial^\beta(x_j - \partial_j V)| \leqslant C_\beta(1+|x|)$, as well as $|\partial^\gamma|\nabla\varphi|^2| \leqslant C_\gamma(1+|x|)^2$, for some constants $C_\beta, C_\gamma > 0$. Let $c := M/(2+3M)$, where $M := \|\nabla V\|_\infty < \infty$. Then $c(1{+}|x|) \leqslant |x|{-}M$ for all $|x| \geqslant 3M/2$. Together with $|\nabla\varphi(x)| = |x{-}\nabla V| \geqslant |x|{-}M$, this implies $|\nabla\varphi(x)| \geqslant c(1{+}|x|)$ for all $x \in \operatorname{supp}(\chi_0)$, if we choose $r = 2M$. By Faá di Bruno's formula, for any $\alpha \in \mathbb{N}^d$, it holds

$$\left|\partial^\alpha\left(\frac{\partial_j\varphi}{|\nabla\varphi|^2}\right)\right| \;\leqslant\; \sum_{|\beta|\leqslant|\alpha|}\sum_{\pi\in\Pi} |\beta|!\binom{\alpha}{\beta}\frac{|\partial^{\alpha-\beta}\partial_j\varphi|\,\prod_{\tau\in\pi}\big|\partial^{|\tau|}|\nabla\varphi|^2\big|}{|\nabla\varphi|^{2(|\pi|+1)}}$$

where $\Pi$ denotes the set of all partitions of $\{1,\ldots,d\}$ and $\tau \in \pi$ means that $\tau$ runs through the elements of the partition $\pi$. Furthermore, $\partial^{|\tau|}$ denotes the partial derivative of order $|\tau|$, given by $\prod_{j\in\tau}\partial_j$. Using the above estimates, this shows

$$\left|\partial^\alpha\left(\frac{\partial_j\varphi}{|\nabla\varphi|^2}\right)(x)\right| \;\leqslant\; \sum_{|\beta|\leqslant|\alpha|}\sum_{\pi\in\Pi} C_{\alpha,\beta,\pi}\frac{(1{+}|x|)(1{+}|x|)^{2|\pi|}}{(1{+}|x|)^{2|\pi|+2}} \;\leqslant\; \frac{C_\alpha}{1+|x|}$$

for all $x \in \operatorname{supp}(\chi_0)$, and some constants $C_\alpha, C_{\alpha,\beta,\pi} > 0$. Since $\operatorname{supp}(\partial^\alpha\chi_0) \subset \operatorname{supp}(\chi_0)$ and $\|\partial^\beta\chi_0\|_\infty < \infty$ for all $\alpha \in \mathbb{N}_0^d$, $(*)$ follows by applying the Leibniz product rule.

Next, we introduce the first order differential operator $L = c - ih\sum_{j=1}^d a_j\partial_j$, where $c$ is given by $c(x) := \chi_1(x) + ih\,\mathrm{div}(a)(x)$ for all $x \in \mathbb{R}^d$, and with domain $\mathcal{D}(L) = \mathcal{S}(\mathbb{R}^d)$. By using integration by parts on $\mathcal{S}(\mathbb{R}^d)$, we find the formal adjoint of $L$ with respect to the $L^2(\mathbb{R}^d)$-norm to be given by $L^* = ih\sum_{j=1}^d a_j\partial_j + \chi_1$. Indeed, for any $f \in C^\infty(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle g, Lf\rangle_2 \;=\; -ih\sum_{j=1}^d \int \overline{g}\,\partial_j(a_j f) + \int \overline{g}\,\chi_1 f = \Big\langle ih\sum_{j=1}^d a_j\partial_j g + \chi_1 g, f\Big\rangle_2$$

From the definition of $a_j$, it follows $\sum_{j=1}^d a_j\partial_j\varphi = 1-\chi_1$, and thus $L^* e^{\frac{i}{h}\varphi} = e^{\frac{i}{h}\varphi}$. Thus, for any $N \in \mathbb{N}$, we obtain

$$I_0(h,\psi_\varepsilon) \;=\; \int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\, L^N\Big(g(1-\chi_1)\psi_\varepsilon\Big)(x)\,dx$$

By direct inspection, we find that for any $u \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $L^N u(x)$ takes the form $\sum_{k=0}^N \sum_{|\alpha_k|=k} \lambda_{\alpha_k}^N(x)\partial^{\alpha_k}u(x)$, where $\lambda_k^N$ is a finite sum of terms, each term being proportional to a multiplication of $N$ functions, where each factor is either given by $\chi_1$, $a_j$, for some $j = 1,\dots,d$ or one of their derivatives. From this, we can see that $\lambda_{\alpha_k}^N(x)\partial^{\alpha_k}(g\chi_0)$ belongs to $L^1(\mathbb{R}^d)$: The terms containing $\chi_1$ or any of its derivatives have compact support and so are in $L^1(\mathbb{R}^d)$. The $N$ factors of each of the remaining terms are therefore of the form $\partial^\alpha a_j$ for some $\alpha \in \mathbb{N}_0^d$ and $j = 1,\dots,d$. Now,

$$I_0(h,\psi_\varepsilon) = \sum_{k=0}^N \sum_{|\alpha_k|=k} \int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\lambda_{\alpha_k}^N(x)\partial^{\alpha_k}(g(1-\chi_1)\psi_\varepsilon)(x)\,dx$$

and by $(*)$ and the assumptions on $g$, there exist constants $c_{N,g}$ and $d_{N,g}$ such that each of the integrands is bounded by

$$c_{N,g}\,\frac{(1+|x|^2)^{p/2}}{(1+|x|)^N} + d_{N,g}\,\chi_1(x) \;\leqslant\; \frac{c_{N,g}}{(1+|x|)^{N-p}} + d_{N,g}\,\chi_1(x)$$

which belongs to $L^1(\mathbb{R}^d)$, whenever $N-p > d$. Hence by choosing $N > d+p$, and by using the theorem of dominated convergence, we see that $\lim_{\varepsilon\to 0} I_0(h,\psi_\varepsilon)$ exists and is independent of $\psi \in \mathcal{S}^*$. More precisely

$$\lim_{\varepsilon\to 0} I_0(h,\psi_\varepsilon) \;=\; \sum_{k=0}^N \sum_{|\alpha_k|=k} \int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\lambda_{\alpha_k}^N(x)\partial^{\alpha_k}(g(1-\chi_1))(x)\,dx \tag{3.9}$$

since from the chain rule, it follows $|\partial^\alpha\psi_\varepsilon(x)| \leqslant \varepsilon^{|\alpha|}\,\|\partial^\alpha\psi\|_\infty$ for any $\alpha \in \mathbb{N}_0^d$, and so, for all $x \in \mathbb{R}^d$, $\lim_{\varepsilon\to 0}\partial^\alpha(f\psi_\varepsilon)(x) = \partial^\alpha f(x)\,\psi(0) = \partial^\alpha f(x)$ for any $f \in C^\infty(\mathbb{R}^d)$. □

Based on Lemma 9, we are now able to provide a Fubini type theorem, which under certain conditions allows to interchange the order of ordinary with oscillatory integrals. The proof is taken from [1, Corollary 1.7].

**Proposition 13** *(Fubini type theorem). Let $m$ be a complex measure on a measurable space $X$, and let $g : X \times \mathbb{R}^d \to \mathbb{C}$, such that for each fixed $\xi \in X$, $g(\xi, \cdot) \in C^\infty(\mathbb{R}^d)$ and $\forall \alpha \in \mathbb{N}_0^d \; \exists \, c_\alpha > 0$ so that $\forall \xi \in X$, $\forall x \in \mathbb{R}^d$,*

$$|\partial_x^\alpha g(\xi, x)| \leqslant c_\alpha (1+|x|^2)^{p/2}$$

*for some $p > 0$, and moreover assume that $g(\cdot, x)$ is bounded for all $x \in \mathbb{R}^d$ (hence it is $m$-integrable). Then the function on $\mathbb{R}^d$, given by $G(x) := \int_X g(\xi, x)\, dm(\xi)$ is Fresnel integrable, and*

$$\int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} G(x)\, dx = \int_X \left( \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} g(\xi, x)\, dx \right) dm(\xi) \tag{3.10}$$

*where, by Lemma 9, $\int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} g(\xi, x)\, dx$ exists for all $\xi \in X$.*

***Proof.*** For any compact set $K \subset \mathbb{R}^d$, $\exists \, C_{\alpha,K} > 0$, $|\partial_x^\alpha g(\xi, x)| \leqslant C_\alpha (1+|x|^2)^{p/2} < C_{\alpha,K}$, $\forall x \in K$ and $\forall \xi \in X$. Since $|m|(X) < \infty$, by the theorem of dominated convergence, it follows $\partial^\alpha G(x) = \int \partial_x^\alpha g(\xi, x) dm(\xi)$, i.e.

$$|\partial^\alpha G(x)| \;\leqslant\; \int_X |\partial_x^\alpha g(\xi, x)|\, d|m|(\xi) \;\leqslant\; \|m\|\, C_\alpha\, (1+|x|^2)^{p/2}$$

Thus, by Lemma 9, $G$ is Fresnel integrable. Choose $\chi_1 \in C_0^\infty(\mathbb{R}^d)$, $\chi_0 \in C^\infty(\mathbb{R}^d)$ as in the proof of Lemma 9, such that

$$I(h) := \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} G(x) dx = I_0(h) + I_1(h)$$

where $I_j(h)$ denotes the Fresnel integral of the product $G\chi_j$, and $I_0(h)$ is given by (3.9) with $g$ replaced by $G$. The same holds for the oscillatoy integral of $g(\xi, \cdot)$, i.e. we write

$$J(\xi, h) := \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} g(\xi, x) dx \;=\; J_0(\xi, h) + J_1(\xi, h)$$

where $J_j(\xi, h)$ is the oscillatory integral of $g(\xi, \cdot)\chi_j$, for $j = 0, 1$, and $J_0(\xi, h)$ can be calculated by using (3.9). In order to show (3.10), we prove the equation separately for both, $I_0$ and $I_1$, i.e.

$$I_j(h) = \int_X J_j(\xi, h)\, dm(\xi)$$

for $j = 0, 1$. In the case $j$=1, this follows from Fubini's theorem, since $I_1(h)$ and $J_1(\xi, h)$ are given by Lebesgue integrals on $\mathbb{R}^d$ (due to $\chi_1 \in C_0^\infty$), and

$$(\xi, x) \mapsto |g(\xi, x)\chi_1(x)|$$

is $m \otimes \lambda^d$ integrable, where $\lambda^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. By (3.9) and Fubini's theorem, which again applies due to the finiteness of $|m|$ and the bound on $\partial_x^\alpha g(\xi, x)$ being uniform in $\xi$, it holds

$$
\begin{aligned}
\int_X I_0(\xi,h)\,dm(\xi) &= \sum_{k\leqslant N}\sum_{|\alpha_k|=k}\int_{\mathbb{R}^d}\int_X e^{\frac{i}{h}\varphi(x)}\lambda^N_{\alpha_k}(x)\partial^{\alpha_k}(g(\xi,\cdot)\chi_0)(x)\,dm(\xi)\,dx \\
&= \sum_{k\leqslant N}\sum_{|\alpha_k|=k}\sum_{\beta\leqslant\alpha_k}\tbinom{\alpha_k}{\beta}\int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\lambda^N_{\alpha_k}(x)\int_X \partial^\beta_x g(\xi,x)\,dm(\xi)\,\partial^{\alpha_k-\beta}\chi_0(x)\,dx \\
&= \sum_{k\leqslant N}\sum_{|\alpha_k|=k}\int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\lambda^N_{\alpha_k}(x)\sum_{\beta\leqslant\alpha_k}\tbinom{\alpha_k}{\beta}\partial^\beta G(x)\,\partial^{\alpha_k-\beta}\chi_0(x)\,dx \\
&= \sum_{k\leqslant N}\sum_{|\alpha_k|=k}\int_{\mathbb{R}^d} e^{\frac{i}{h}\varphi(x)}\lambda^N_{\alpha_k}(x)\,\partial^\alpha_k(G\chi_0)(x)\,dx \;=\; I_0(h)
\end{aligned}
$$

where we have used the Leibniz product rule, equation (3.9), and the result from above, which allows to differentiate $G$ under the integral sign. □

Our following definition of Hermite polynomials is taken from [1, 4.18], whereas the proofs of Lemma 10 and Proposition 14 are based on [1, Proposition A.1], but were rewritten due to many missing steps.

**Lemma 10** *(Hermite polynomials). For any $n \in \mathbb{N}$, $h \in \mathbb{R}$, and $b_1,\dots,b_n \in \mathbb{R}^d$, we define the Hermite polynomial of order $n$, $H_n(b_1,\dots,b_n|h)$, by*

$$H_n(b_1,\dots,b_n|h)(x) \;:= e^{\frac{ih}{2}|x|^2}\Big(\mathcal{D}_{b_1}\cdots\mathcal{D}_{b_n}e^{-\frac{ih}{2}|\cdot|^2}\Big)(x)$$

*where for any $b \in \mathbb{R}^d$, $\mathcal{D}_b$ denotes the directional derivative in the direction of $b$, which for any $f \in C^1(\mathbb{R}^d)$ is given by $(\mathcal{D}_b f)(x) = \langle\nabla f(x), b\rangle_{\mathbb{R}^d}$. Then for each $n \in \mathbb{N}$,*

$$H_{2n}(b_1,\dots,b_{2n}|h)(x) \;=\; \sum_{m=0}^{n}(-ih)^{2n-m}\,\alpha^m_{b_1,\dots,b_{2n}}(x) \tag{3.11}$$

*where $|\alpha^m_{b_1,\dots,b_{2n}}(x)| \leqslant (2n)!/((2n-2m)!\,m!\,2^m)|b_1|\cdots|b_{2n}|\,|x|^{2n-2m}$ for all $m \leqslant n$.*

***Proof.*** We will show, that *for each $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $H_{2n}(b_1,\dots,b_{2n}|h)(x)$ takes the form (3.11), where $\alpha^m_{b_1,\dots,b_{2n}}(x)$ is a sum of $(2n)!/((2n-2m)!\,m!\,2^m)$ terms, each term being a product of $2n-m$ factors, where each factor is either of the form $\langle b_i,b_j\rangle_{\mathbb{R}^d}$ or $\langle x,b_k\rangle_{\mathbb{R}^d}$ for some $i,j,k \in \{1,\dots,2n\}$, but in each product every $b_i$ appears only once while $x$ appears $2n-2m$ times.* Hence there are $m$ factors of the form $\langle b_i,b_j\rangle$ and $2n-2m$ of the form $\langle x,b_k\rangle$. The estimate on $\alpha^m_{b_1,\dots,b_{2n}}$ then follows from the Cauchy-Schwarz inequality. Since $\big\langle(\nabla e^{-\frac{ih}{2}|\cdot|^2})(x), b\big\rangle_{\mathbb{R}^d} = -ih\,e^{-\frac{ih}{2}|x|^2}\langle x,b\rangle_{\mathbb{R}^d}$, for any $x, b \in \mathbb{R}^d$, we have $H_1(b_1|h)(x) = -ih\,\langle x,b_1\rangle$, and therefore

$$H_2(b_1,b_2|h)(x) \;=\; -ih\,\langle b_1,b_2\rangle + (-ih)^2\,\langle x,b_1\rangle\langle x,b_2\rangle$$

Hence the assertion is true in the case $n = 1$. By induction, assume that it is true for some $n \in \mathbb{N}$. From $\partial_j \partial_k f = \partial_k \partial_j f$ for any $f \in C^\infty(\mathbb{R}^d)$ and $j, k = 1, \dots, d$, it follows $\mathcal{D}_{b_1} \mathcal{D}_{b_2} f = \mathcal{D}_{b_2} \mathcal{D}_{b_1} f$ for any $b_1, b_2 \in \mathbb{R}^d$. Thus, we have

$$\begin{aligned} H_{2(n+1)}(b_1, \dots, b_{2(n+1)}|h) &= e^{\frac{ih}{2}|\cdot|^2} \mathcal{D}_{b_{2n+2}} \mathcal{D}_{b_{2n+1}} \Big( e^{-\frac{ih}{2}|\cdot|^2} H_{2n}(b_1, \cdots, b_{2n}|h) \Big) \\ &= H_2(b_{2n+1}, b_{2n+2}|h)\, H_{2n}(b_1, \dots, b_{2n}|h) + H_1(b_{2n+1}|h)\, \mathcal{D}_{b_{2n+2}} H_{2n}(b_1, \dots, b_{2n}) \\ &\quad + H_1(b_{2n+2}|h)\, \mathcal{D}_{b_{2n+1}} H_{2n}(b_1, \dots, b_{2n}|h) + \mathcal{D}_{b_{2n+1}} \mathcal{D}_{b_{2n+2}} H_{2n}(b_1, \dots, b_{2n}|h) \end{aligned}$$

The terms in the sum $\alpha^m_{b_1,\dots,b_{2n}}(x)$ are of the form

$$\langle b_{\sigma(1)}, x\rangle \cdots \langle b_{\sigma(2n-2m)}, x\rangle \langle b_{\sigma(2n-2m+1)}, b_{\sigma(2n-2m+2)}\rangle \cdots \langle b_{\sigma(2n-1)}, b_{\sigma(2n)}\rangle$$

where $\sigma$ is a permutation of $2n$ elements. Hence, for $m = n$, any derivative of $\alpha^m_{b_1,\dots,b_{2n}}$ vanishes, whereas for $m < n$, $\nabla \alpha^m_{b_1,\dots,b_{2n}}(x)$ consists of terms

$$\langle b_{\sigma(2n-2m+1)}, b_{\sigma(2n-2m+2)}\rangle \cdots \langle b_{\sigma(2n-1)}, b_{\sigma(2n)}\rangle \sum_{j=1}^{2n-2m} \Big( \prod_{i \neq j} \langle b_{\sigma(i)}, x\rangle \Big)\, b_{\sigma(j)}$$

and therefore $\mathcal{D}_b H_{2n}(b_1, \dots, b_{2n}|h)$ takes the form of (3.11), with $\alpha^m_{b_1,\dots,b_{2n}}$ replaced by $\beta^m_{b_1,\dots,b_{2n},b}$ which, for $m < n$ is a sum of $(2n)!(2n-2m)/((2n-2m)!\, m!\, 2^m)$ terms, such that each term contains $2n-2m-1$ factors of the form $\langle b_i, x\rangle$, as well as $m$ factors of the form $\langle b_j, b_k\rangle$, and one factor $\langle b_l, b\rangle$, where $i, j, k, l \in \{1, \dots, 2n\}$, and in every term, each $b_i$ only appears once. Moreover, $\beta^n_{b_1,\dots,b_{2n},b} \equiv 0$. Also, for any $b, b' \in \mathbb{R}^d$, the expression $\mathcal{D}_{b'} \mathcal{D}_b H_{2n}(b_1, \dots, b_{2n}|h)$ takes the form of (3.11), but with $\alpha^m_{b_1,\dots,b_{2n}}$ replaced by $\gamma^m_{b_1,\dots,b_{2n},b,b'}$ which for $m < n$ is a sum of $(2n)!(2n-2m-1)(2n-2m)/((2n-2m)!\, m!\, 2^m)$ terms, where each term contains $2n-2m-2$ factors of the form $\langle b_i, x\rangle$, $m$ factors of the form $\langle b_j, b_k\rangle$, one factor of the form $\langle b_l, b\rangle$ and one factor $\langle b_r, b'\rangle$, where $i, j, k, l, r \in \{1, \dots, 2n\}$, and moreover $\gamma^n_{b_1,\dots,b_{2n},b,b'} \equiv 0$. It follows

$$\begin{aligned} H_{2(n+1)}(b_1, \dots, b_{2n+2}|h) &= \sum_{m=0}^{n} (-ih)^{2n-m+2} \langle x, b_{2n+1}\rangle \langle x, b_{2n+2}\rangle\, \alpha^m_{b_1,\dots,b_{2n}} \\ &\quad + \sum_{m=0}^{n} (-ih)^{2n-m+1} \big( \langle x, b_{2n+1}\rangle\, \beta^m_{b_1,\dots,b_{2n},b_{2n+2}} + \langle x, b_{2n+2}\rangle\, \beta^m_{b_1,\dots,b_{2n},b_{2n+1}} \big) \\ &\quad + \sum_{m=0}^{n} (-ih)^{2n-m+1} \langle b_{2n+1}, b_{2n+2}\rangle\, \alpha^m_{b_1,\dots,b_{2n}} + \sum_{m=0}^{n} (-ih)^{2n-m}\, \gamma^m_{b_1,\dots,b_{2n+2}} \\ &= \sum_{m=0}^{n} (-ih)^{2n-m+2} \langle x, b_{2n+1}\rangle \langle x, b_{2n+2}\rangle\, \alpha^m_{b_1,\dots,b_{2n}} \\ &\quad + \sum_{m=1}^{n} (-ih)^{2n-m+2} \big( \langle x, b_{2n+1}\rangle\, \beta^{m-1}_{b_1,\dots,b_{2n},b_{2n+2}} + \langle x, b_{2n+2}\rangle\, \beta^{m-1}_{b_1,\dots,b_{2n},b_{2n+1}} \big) \\ &\quad + \sum_{m=1}^{n} (-ih)^{2n-m+2} \langle b_{2n+1}, b_{2n+2}\rangle\, \alpha^{m-1}_{b_1,\dots,b_{2n}} + \sum_{m=2}^{n+1} (-ih)^{2n-m+2}\, \gamma^{m-2}_{b_1,\dots,b_{2n+2}} \end{aligned}$$

Hence $H_{2(n+1)}(b_1,\ldots,b_{2n+2}|h)$ is of the form $\sum_{m=0}^{n+1}(-ih)^{2n-m+2}\,\delta^m_{b_1,\ldots,b_{2n+2}}$, with

$$\begin{aligned}
\delta^0_{b_1,\ldots,b_{2n+2}} &= \langle x,b_{2n+1}\rangle\langle x,b_{2n+2}\rangle\,\alpha^0_{b_1,\ldots,b_{2n}}\\
&= \langle x,b_1\rangle\cdots\langle x,b_{2n}\rangle\langle x,b_{2n+1}\rangle\langle x,b_{2n+2}\rangle\\
\delta^1_{b_1,\ldots,b_{2n+2}} &= \langle x,b_{2n+1}\rangle\langle x,b_{2n+2}\rangle\,\alpha^1_{b_1,\ldots,b_{2n}} + \langle x,b_{2n+1}\rangle\,\beta^0_{b_1,\ldots,b_{2n},b_{2n+2}}\\
&\quad + \langle x,b_{2n+2}\rangle\,\beta^0_{b_1,\ldots,b_{2n},b_{2n+1}} + \langle b_{2n+1},b_{2n+2}\rangle\,\alpha^0_{b_1,\ldots,b_{2n}}\\
\delta^m_{b_1,\ldots,b_{2n+2}} &= \langle x,b_{2n+1}\rangle\langle x,b_{2n+2}\rangle\,\alpha^m_{b_1,\ldots,b_{2n}} + \langle x,b_{2n+1}\rangle\,\beta^{m-1}_{b_1,\ldots,b_{2n},b_{2n+2}}\\
&\quad + \langle x,b_{2n+2}\rangle\,\beta^{m-1}_{b_1,\ldots,b_{2n},b_{2n+1}} + \langle b_{2n+1},b_{2n+2}\rangle\,\alpha^{m-1}_{b_1,\ldots,b_{2n}}\\
&\quad + \gamma^{m-2}_{b_1,\ldots,b_{2n+2}} \qquad\qquad (m\geqslant 2)
\end{aligned}$$

So $\delta^1_{b_1,\ldots,b_{2n+2}}$ is a sum of $n(2n-1)+4n+1 = (2n+2)!/((2n)!\,2) =: A_{n,1}$ terms, where $A_{n,m} := (2n+2)!/((2n+2-2m)!\,m!\,2^m)$. Moreover, for $m\geqslant 2$, $\delta^m_{b_1,\ldots,b_{2n+2}}$ is a sum of

$$\frac{A_{n,m}}{(2n+2)(2n+1)}\Big((2n-2m+2)(2n-2m+1)+4m(2n-2m+2)+2m+4m(m-1)\Big) = A_{n,m}$$

terms. Also, each term consists of $2n+2-m$ factors, with $m$ factors of the form $\langle b_i,b_j\rangle$ and $2n+2-2m$ factors of the form $\langle x,b_k\rangle$, where $i,j,k\in\{1,\ldots,2n+2\}$ and each $b_i$ appears only once. This shows that the assertion from the beginning of the proof is true for $n+1$, under the assumption that it holds for $n$. By induction, the claim is true for all $n\in\mathbb{N}$. □

**Proposition 14**. *For any $n\in\mathbb{N}$, $b_1,\ldots,b_n\in\mathbb{R}^d$, $h\in\mathbb{R}$, it holds*

$$H_n(b_1,\ldots,b_n|h)(x) = \sum_{m=0}^{\lfloor n/2\rfloor}(-ih)^{n-m}\,\alpha^m_{b_1,\ldots,b_n}(x) \tag{3.12}$$

*where $|\alpha^m_{b_1,\ldots,b_n}(x)|\leqslant n!/((n-2m)!\,m!\,2^m)|b_1|\cdots|b_n|\,|x|^{n-2m}$ for all $m\leqslant\lfloor n/2\rfloor$.*

***Proof.*** For $n$ even, the assertion coincides with Lemma 10. For $n=2k+1$, $k\in\mathbb{N}$,

$$\begin{aligned}
H_{2k+1}(b_1,\ldots,b_{2k+1}|h)(x) &= e^{\frac{ih}{2}|x|^2}\mathcal{D}_{b_{2k+1}}\big(e^{-\frac{ih}{2}|\cdot|^2}H_{2k}(b_1,\ldots,b_{2k}|h)\big)(x)\\
&= H_1(b_{2k+1}|h)(x)\,H_{2k}(b_1,\ldots,b_{2k}|h)(x)+\mathcal{D}_{b_{2k+1}}H_{2k}(b_1,\ldots,b_{2k}|h)(x)\\
&= \sum_{m=0}^{k}(-ih)^{2k+1-m}\langle x,b_{2k+1}\rangle\,\alpha^m_{b_1,\ldots,b_{2k}}(x)+\sum_{m=0}^{k-1}(-ih)^{2k-m}\beta^m_{b_1,\ldots,b_{2k+1}}(x)
\end{aligned}$$

where, for $m<k$, $\beta^m_{b_1,\ldots,b_{2k+1}}$ is a sum of $(2k)!(2k-2m)/((2k-2m)!\,m!\,2^m)$ terms, such that each term contains $2k-2m-1$ factors of the form $\langle b_i,x\rangle$, as well as $m$ factors of

the form $\langle b_j, b_l\rangle$, and one factor $\langle b_s, b\rangle$, where $i, j, l, s \in \{1, \ldots, 2k\}$, and in every term, each $b_i$ only appears once. We obtain

$$H_{2k+1}(b_1, \ldots, b_{2k+1}|h) \;=\; \sum_{m=0}^{k} (-ih)^{2k+1-m}\, \kappa^m_{b_1,\ldots,b_{2k+1}}$$

where

$$\begin{aligned} \kappa^0_{b_1,\ldots,b_{2k+1}} &= \langle x, b_{2k+1}\rangle\, \alpha^0_{b_1,\ldots,b_{2k}} \;=\; \langle x, b_1\rangle \cdots \langle x, b_{2k+1}\rangle \\ \kappa^m_{b_1,\ldots,b_{2k+1}} &= \langle x, b_{2k+1}\rangle\, \alpha^m_{b_1,\ldots,b_{2k}} + \beta^{m-1}_{b_1,\ldots,b_{2k+1}} \end{aligned}$$

for $1 \leqslant m \leqslant k$. Hence, $\kappa^m_{b_1,\ldots,b_{2k+1}}$ is a sum of

$$\frac{(2k)!}{(2k-2m)!\, m!\, 2^m} + \frac{(2k)!(2k-2m+2)}{(2k-2m+2)!\,(m-1)!\, 2^{m-1}}$$

$$= \left(\frac{2k+1-2m}{2k+1} + 2m\frac{2k-2m+2}{(2k+2-2m)(2k+1)}\right)\frac{(2k+1)!}{(2k+1-2m)!\, m!\, 2^m} \;=\; \frac{n!}{(n-2m)!\, m!\, 2^m}$$

terms, each term being a product of $2n+1-m$ factors, where each factor is either of the form $\langle b_i, b_j\rangle$ or $\langle x, b_l\rangle$ for some $i, j, l \in \{1, \ldots, 2k+1\}$, but in each product every $b_i$ appears only once, while $x$ appears $2k+1-2m = n-2m$ times. Since $\lfloor n/2\rfloor = k$, the estimate follows from the Cauchy-Schwarz inequality, as in the proof of Lemma 10. □

As an application of the Fubini type theorem (Proposition 13) and the preceding results, the next Lemma shows how to calculate oscillatory integral of functions on $\mathbb{R}^d$ of the form

$$f(x) \;=\; e^{i\langle x, b_0\rangle}\langle x, b_1\rangle \cdots \langle x, b_k\rangle$$

where $b_0, \ldots, b_k \in \mathbb{R}^d$. This result was indicated in [1, Lemma 4.4], but the proof provided there needs some more care, which is why we have slightly changed the statement and added some crutial steps to the proof.

**Lemma 11**. *For any $b_0, b_1, \ldots, b_k \in \mathbb{R}^d$, where $k \in \mathbb{N}$, $x \mapsto e^{i\langle x, b_0\rangle}\langle x, b_1\rangle \cdots \langle x, b_k\rangle$, is Fresnel integrable, and it holds*

$$\int^o_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} e^{i\langle x, b_0\rangle}\langle x, b_1\rangle \cdots \langle x, b_k\rangle\, dx \;=\; \lambda^k_h\, e^{-\frac{ih}{2}|b_0|^2} H_k(b_1, \ldots, b_k|h)(b_0) \tag{3.13}$$

*where $\lambda^k_h = (-i)^k (2\pi i h)^{\frac{d}{2}}$.*

***Proof.*** Fresnel integrability already follows from Lemma 9 applied to the case $V \equiv 0$. For (3.13), choose $\psi \in \mathcal{S}(\mathbb{R}^d)$, such that $\hat\psi \in \mathcal{S}^*(\mathbb{R}^d)$, i.e. $\hat\psi(0) = 1$. Then $\hat\psi(\varepsilon x) = \hat\psi_\varepsilon$, where $\psi_\varepsilon(y) := \varepsilon^{-d}\psi(\frac{y}{\varepsilon})$. Since, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $b \in \mathbb{R}^d$, $\widehat{\mathcal{D}_b\varphi}(x) = -i\langle x, b\rangle\, \hat\varphi(x)$, it holds

$$\mathcal{F}(\mathcal{D}_{b_1} \cdots \mathcal{D}_{b_k}\psi_\varepsilon)(x) \;=\; (-i)^k\langle x, b_1\rangle \cdots \langle x, b_k\rangle\, \hat\psi_\varepsilon$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$. Hence, from Lemma 1 and equation

(2.11), it follows

$$\begin{aligned}\int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} e^{i\langle x,b_0\rangle}\langle x,b_1\rangle\cdots\langle x,b_k\rangle\,\hat\psi(\varepsilon x)\,dx &= i^k\int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2}\mathcal{F}(\delta_{b_0}*\mathcal{D}_{b_1}\cdots\mathcal{D}_{b_k}\psi_\varepsilon)\,dx\\ &= i^k(2\pi i h)^{d/2}\int_{\mathbb{R}^d} e^{-\frac{ih}{2}|x+b_0|^2}(\mathcal{D}_{b_1}\cdots\mathcal{D}_{b_k}\psi_\varepsilon)(x)\,dx\\ &= (-i)^k(2\pi i h)^{d/2}\int_{\mathbb{R}^d}\mathcal{D}_{b_1}\cdots\mathcal{D}_{b_k}e^{-\frac{ih}{2}|x|^2}\,\psi_\varepsilon(x-b_0)\,dx\\ &= \lambda_h^k\int_{\mathbb{R}^d} e^{-\frac{ih}{2}|x|^2}H_k(b_1,\dots,b_k|h)(x)\,\psi_\varepsilon(x-b_0)\,dx\end{aligned}$$

Let $B_1$ be the unit ball in $\mathbb{R}^d$, then for any $g\in C(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} g(\varepsilon x)\psi(x)\,dx \;=\; \int_{B_1} g(\varepsilon x)\psi(x)\,dx + \int_{\mathbb{R}^d\setminus B_1} g(\varepsilon x)\psi(x)\,dx$$

where the first term converges to $g(0)\int\psi(x)dx = g(0)$, by dominated convergence, the continuity of $g$, and $\int\psi(x)dx=\hat\psi(0)=1$. If we assume that $g$ is at most of polynomial growth, i.e. if $|g(x)|\leqslant p_N(x)$, for some polynomial $p_N$ of order $N\in\mathbb{N}$, then after a change of variables, we obtain for the absolute value of the second term

$$\Big|\int_{\mathbb{R}^d}\mathbb{1}_{|y|\geqslant\varepsilon}(y)\,g(y)\,\varepsilon^{-d}\,\psi\Big(\frac{y}{\varepsilon}\Big)dy\Big| \;\leqslant\; C_\psi\int_{\mathbb{R}^d}\frac{|g(y)|}{(1+|\frac{y}{\varepsilon}|)^M}|\varepsilon|^{-d}\,dy$$

where $M := N+d+1$ and $C_\psi := \sup_{x\in\mathbb{R}^d}(1+|x|)^M|\psi(x)| < \infty$, since $\psi\in\mathcal{S}(\mathbb{R}^d)$. By the assumption on $g$ and the choice of $M$, the integral on the right-hand side exists, and moreover its integrand is monotone decreasing to 0, as $\varepsilon\to 0$. Therefore, by monotone convergence, $\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} g(\varepsilon x)\psi(x)\,dx = 0$. Using this result in the calculation above, where $g(x)$ is given by $e^{-\frac{ih}{2}|x+b_0|^2}H_k(b_1,\dots,b_k|h)(x+b_0)$, together with $\int_{\mathbb{R}^d} g(\varepsilon x)\psi(x)dx = \int_{\mathbb{R}^d} g(x)\psi_\varepsilon(x)dx$, shows that

$$\int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2}e^{i\langle x,b_0\rangle}\langle x,b_1\rangle\cdots\langle x,b_k\rangle\hat\psi(\varepsilon x)\,dx \;\xrightarrow{\varepsilon\to 0}\; \lambda_h^k\,e^{-\frac{ih}{2}|b_0|^2}H_k(b_1,\dots,b_k|h)(b_0)$$

which proves the claim, since we know that the limit is independent of $\hat\psi\in\mathcal{S}^*$. □

Having done most of the preparations for the proof of Theorem 4, we are left with some minor technicalities provided by the following two Lemmas. The proofs consist of basic calculations.

**Lemma 12** *(Translation Principle). If $f{=}\hat\mu\in\mathcal{F}(\mathcal{H})$, where $\mathcal{H}$ is real separable Hilbert space, then for any $a\in\mathcal{H}$, it holds*

$$\mathcal{F}^h_{\mathcal{H}}\big(x\mapsto f(x{+}a)\,e^{\frac{i}{h}\langle x,a\rangle}\big) \;=\; e^{-\frac{i}{2h}\|a\|^2}\mathcal{F}^h_{\mathcal{H}}(f) \tag{3.14}$$

***Proof.*** Let $\mu^a$ be the complex measure on $\mathcal{H}$ with $\mu$-density $y \mapsto e^{i\langle a,y\rangle}$, which means that $d\mu^a(y) = e^{i\langle a,y\rangle}d\mu(y)$, then $f(x+a) = \hat{\mu}^a(x)$. From Lemma 1 and Theorem 2, we obtain

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}\|x\|^2} f(x+a)\, e^{\frac{i}{h}\langle x,a\rangle} dx \;=\; \int_{\mathcal{H}} e^{-\frac{ih}{2}\|x\|^2} d(\mu^a * \delta_{a/h})(x)$$

The claim now follows from $e^{-\frac{ih}{2}\|x+\frac{a}{h}\|^2} e^{i\langle a,x\rangle} = e^{-\frac{i}{2h}\|a\|^2} e^{-\frac{ih}{2}\|x\|^2}$ and another application of Theorem 2. □

**Lemma 13**. *If, for any $k\in\mathbb{N}$, we set $e_k(t) := \sum_{j\geqslant k} \frac{(-it)^j}{j!}$, $t\in\mathbb{R}$, then $|e_k(t)| \leqslant \frac{|t|^k}{k!}$.*

***Proof.*** Let us first show that for any $k \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$e_k(z) \;=\; \frac{z^k}{(k-1)!} \int_0^1 (1-t)^{k-1} e^{tz} dt$$

For $k = 1$, this just follows from $\int_0^1 e^{-tz}dt = (e^z - 1)/z$. Now, assume that the assertion holds for $k$, then

$$\begin{aligned}\sum_{j\geqslant k+1} \frac{z^j}{j!} \;&=\; \frac{z^k}{(k-1)!}\int_0^1 (1-t)^{k-1} e^{tz}\, dt - \frac{z^k}{k!} \;=\; \frac{z^k}{k!}\Big( - \int_0^1 \frac{d}{dt}(1-t)^k e^{tz} dt - 1\Big)\\ &=\; \frac{z^k}{k!}\Big(1 - \int_0^1 (1-t)^k z e^{tz} dt - 1\big) \;=\; \frac{z^{k+1}}{k!}\int_0^1 (1-t)^k e^{tz} dt\end{aligned}$$

By induction, this proves the given identity. Since $\int_0^1 (1-t)^{k-1} dt = -1/k$, this shows the estimate. □

The overall strategy for the proof of Theorem 4 is taken from [1] and [29]. We have added justifications and explicit calculations where they could have been considered missing.

***Proof of Theorem 4.*** From (3.5), it follows $\int_{\mathcal{H}} \|x\|^2 d|\mu|(x) \leqslant 2C_\mu/\varepsilon^2 < 1$, and therefore, Lemma 7 implies that $\exists!\, a \in \mathcal{H}$, such that $DV(a) = \langle a, \cdot\rangle$. For $\phi_0 := \|\cdot\|^2$, it holds $\phi_0(x+h) - \phi_0(x) - 2\langle x, h\rangle = \|h\|^2$, and therefore $D\phi_0(x) = 2\langle x, \cdot\rangle$. Thus, it holds $D\phi(a) = \langle a, \cdot\rangle - DV(a) = 0$, i.e. $\phi$ admits a unique stationary point given by $a$.

By using the Translational Principle (Lemma 12), we can write $I(h) = e^{\frac{i}{h}\phi(a)} I^*(h)$, where for $g_a(x) := g(x+a)$ we have set

$$I^*(h) := \mathcal{F}^h_{\mathcal{H}}\big(x \mapsto e^{-\frac{i}{h}(V(x+a) - V(a) - \langle x,a\rangle)} g_a(x)\big)$$

For $h \in \mathbb{R}$, let $(I_q(h))_{q\in\mathbb{N}}$ be a sequence of finite-dimensional oscillatory Fresnel integrals such that $I^*(h) = \lim_{q\to\infty} I_q(h)$, and $I_q(h) = \mathcal{F}^h_{\mathcal{H}_q}\big(e^{-\frac{i}{h}W|_{\mathcal{H}_q}} g_a\big|_{\mathcal{H}_q}\big)$, where $W$ is given by $W(x) := V_a(x) - V(a) - \langle a, x\rangle$, $V_a(x) := V(x+a)$. From integration by parts,

$$\zeta^2 \int_0^1 (1-t)\, e^{\zeta t}\, dt \;=\; \zeta \int_0^1 (1-t)\, \frac{d}{dt} e^{\zeta t}\, dt \;=\; e^{\zeta} - \zeta - 1 \;=\; \sum_{k \geqslant 2} \frac{\zeta^k}{k!}$$

for any $\zeta \in \mathbb{R}$. Since $V_a(x) - V(a) = \int (e^{i\langle x,y\rangle} - 1)\, d\mu^a(y)$, it follows

$$V_a(x) - V(a) = \int_{\mathcal{H}} (i\langle x,\alpha\rangle)^2 \int_0^1 (1-t) e^{i\langle x,\alpha\rangle t}\, dt\, d\mu^a(\alpha) + i \int_{\mathcal{H}} \langle \alpha, x\rangle d\mu^a(\alpha)$$

From Proposition 10, $DV(x) = i \int \langle \alpha, \cdot\rangle d\mu^a(\alpha)$, and therefore, $DV(a) = \langle a, \cdot\rangle$ implies

$$W(x) \;=\; -\int_{\mathcal{H}} \int_0^1 (1-t)\, \langle x,\alpha\rangle^2\, e^{i\langle x,\alpha\rangle t}\, dt\, d\mu^a(\alpha)$$

For fixed $q \in \mathbb{N}$, as in the proof of Theorem 2, there exist $\mu_d, \nu_d \in \mathcal{M}(\mathbb{R}^d)$, $d := \dim(\mathcal{H}_q)$, such that $V_a|_{\mathcal{H}_q} \circ \gamma_E^{-1} = \hat{\mu}_d$ and $g_a|_{\mathcal{H}_q} \circ \gamma_E^{-1} = \hat{\nu}_d$, where $\gamma_E : \mathcal{H}_q \to \mathbb{R}^d$ is an arbitrary basis representation of $\mathcal{H}_q$ in $\mathbb{R}^d$. In particular, $\mu_d$ is given by the complex image measure $\mu^a \circ \mathcal{P}_q^{-1} \circ \gamma_E^{-1}$. Then, by definition,

$$I_q(h) \;=\; (2\pi i h)^{-d/2} \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} e^{-\frac{i}{h} w(x)}\, \hat{\nu}_d(x)\, dx \;=:\; J(h)$$

where $w := W|_{\mathcal{H}_q} \circ \gamma_E^{-1}$. We have for any $y \in \mathcal{H}_q$

$$\begin{aligned} W\big|_{\mathcal{H}_q}(y) \;&=\; W(\mathcal{P}_q y) \;=\; -\int_{\mathcal{H}} \int_0^1 (1-t)\, \langle y, \mathcal{P}_q \alpha\rangle^2\, e^{i\langle y, \mathcal{P}_q\alpha\rangle t}\, dt\, d\mu^a(\alpha) \\ &=\; -\int_{\mathcal{H}_q} \int_0^1 (1-t)\, \langle y, \tilde{\alpha}\rangle^2\, e^{i\langle y,\tilde{\alpha}\rangle t}\, dt\, d(\mu^a \circ \mathcal{P}_q^{-1})(\tilde{\alpha}) \end{aligned}$$

Hence, for any $x \in \mathbb{R}^d$, $w(x) = -\int_{\mathbb{R}^d} \int_0^1 (1-t)\, \langle x,y\rangle^2\, e^{i\langle x,y\rangle t}\, dt\, d\mu_d(y)$. If $h \in \mathbb{R}$ is fixed, then by dominated convergence, we have $J(h) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} J_{\varepsilon,N}$, where for any $\varphi \in \mathcal{S}^*$, we have set

$$J_{\varepsilon,N} := (2\pi i h)^{-d/2} \sum_{k=0}^{N} \frac{(-i/h)^k}{k!} \int_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} w(x)^k\, \hat{\nu}_d(x)\, \varphi(\varepsilon x)\, dx$$

The convergence in $\varepsilon$ is uniform in $N$, which allows the limits to be interchanged. Thus,

$$J(h) \;=\; \lim_{N\to\infty} (2\pi i h)^{-d/2} \sum_{n=0}^{N} \frac{(-i/h)^n}{n!} \int_{\mathbb{R}^d}^{o} e^{\frac{i}{2h}|x|^2} \varphi_n(x)\, dx \qquad (i)$$

where $\varphi_n(x) := w(x)^n\, \hat{\nu}_d(x)$, for all $n \in \mathbb{N}$. From the assumptions on $\mu$, from $|\mu^a| = |\mu|$, and $|\gamma_E \mathcal{P}_n \alpha| = \|\mathcal{P}_n \alpha\| \leqslant \|\alpha\|$, it follows $\int |x|^k d|\mu_d|(x) \leqslant \int \|\alpha\|^k d|\mu|(\alpha) < \infty$, $\forall\, k \in \mathbb{N}$.

Thus, we are allowed to apply Fubini's theorem in

$$\begin{aligned}\varphi_n(x) \;&=\; (-1)^n \int_{\mathbb{R}^d}\int_0^1 \cdots \int_{\mathbb{R}^d}\int_0^1 (1-t_1)\cdots(1-t_n)\langle x,\alpha_1\rangle^2\cdots\langle x,\alpha_n\rangle^2 \times \\ &\qquad\times e^{i\langle x,\alpha_1\rangle t_1}\cdots e^{i\langle x,\alpha_n\rangle t_n}\, dt_1\, d\mu_d(\alpha_1)\cdots dt_n\, d\mu_d(\alpha_n)\,\hat{\nu}_d(x) \\ &=\; (-1)^n \int_{\mathbb{R}^d\times([0,1]\times\mathbb{R}^d)^n} (1-t_1)\cdots(1-t_n)\langle x,\alpha_1\rangle^2\cdots\langle x,\alpha_n\rangle^2 e^{i\langle x,f_n(\xi)\rangle}\, d\Gamma_n(\xi)\end{aligned}$$

where $\xi = (\beta,(t_1,\alpha_1),\ldots,(t_n,\alpha_n))$, $f_n(\xi) = \sum_{j=1}^n \alpha_j t_j + \beta$, $\Gamma_n := \nu_d \otimes (\lambda_{[0,1]}\otimes\mu_d)^{\otimes n}$, with $\lambda_{[0,1]}$ denoting the Lebesgue measure on $[0,1]$. If we denote $\mathbb{R}^d\times([0,1]\times\mathbb{R}^d)^n$ by $X$, then from Proposition 13 and Lemma 11 it follows

$$\begin{aligned}&\int^o_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2}\varphi_n(x)\,dx \\ &= (-1)^n \int_X (1-t_1)\cdots(1-t_n)\int^o_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2}\langle x,\alpha_1\rangle^2\cdots\langle x,\alpha_n\rangle^2 e^{i\langle x,f_n(\xi)\rangle}\,dx\,d\Gamma_n(\xi) \\ &= (2\pi i h)^{d/2}\int_X (1-t_1)\cdots(1-t_n)\, e^{-\frac{ih}{2}|f_n(\xi)|^2} H_{2n}\big(\alpha_1,\alpha_1,\ldots,\alpha_n,\alpha_n|h\big)\big(f_n(\xi)\big)\,d\Gamma_n(\xi) \\ &= \sum_{m=0}^n (-ih)^{2n-m}(2\pi i h)^{d/2}\int_X (1-t_1)\cdots(1-t_n)\, e^{-\frac{ih}{2}|f_n(\xi)|^2}\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\,d\Gamma_n(\xi)\end{aligned}$$

where $\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))$ satisfies

$$\big|\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\big| \;\leqslant\; \frac{(2n)!}{(2n-2m)!\,m!\,2^m}\,|\alpha_1|^2\cdots|\alpha_n|^2\,|f_n(\xi)|^{2n-2m} \qquad (ii)$$

Due to $(i)$, we can write $J(h) = \sum_{n=0}^\infty J_n(h)$, with

$$J_n(h) := \sum_{m=0}^n \frac{(-1)^n}{n!}(-ih)^{n-m}\int_X (1-t_1)\cdots(1-t_n)\, e^{-\frac{ih}{2}|f_n(\xi)|^2}\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\,d\Gamma_n(\xi)$$

From $|e^{-ix}-1| \leqslant |x|$ for any $x\in\mathbb{R}$, which we have shown earlier, and

$$J_n(0) \;=\; \frac{(-1)^n}{n!}\int_X (1-t_1)\cdots(1-t_n)\,\gamma^n_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\,d\Gamma_n(\xi)$$

it follows

$$\begin{aligned}|J_n(h)-J_n(0)| \;&\leqslant\; \frac{1}{n!}\sum_{m=0}^{n-1}|h|^{n-m}\int_X (1-t_1)\cdots(1-t_n)\,\big|\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\big|\,d|\Gamma_n|(\xi) \\ &+ \frac{1}{n!}\sum_{m=0}^n |h|^{n+1-m}\int_X (1-t_1)\cdots(1-t_n)\,\frac{|f_n(\xi)|^2}{2}\,\big|\gamma^m_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\big|\,d|\Gamma_n|(\xi) \\ &= \frac{|h|^{n+1}}{n!}\int_X (1-t_1)\cdots(1-t_n)\,\frac{|f_n(\xi)|^2}{2}\,\big|\gamma^0_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\big|\,d|\Gamma_n|(\xi) \\ &+ \frac{1}{n!}\sum_{m=0}^{n-1}|h|^{n-m}\int_X (1-t_1)\cdots(1-t_n)\Big(|\gamma^m_{\alpha_1,\ldots,\alpha_n}| + \frac{|f_n(\xi)|^2}{2}|\gamma^{m+1}_{\alpha_1,\ldots,\alpha_n}|\Big)\,d|\Gamma_n|(\xi)\end{aligned}$$

If we define

$$\omega_{n,N} := \int_X (1-t_1)\cdots(1-t_n)\,|\alpha_1|^2\cdots|\alpha_n|^2\,|f_n(\xi)|^N\,d|\Gamma_n|(\xi)$$

then for $|h|<1$, $(ii)$ implies

$$\begin{aligned}|J_n(h)-J_n(0)| \leqslant |h| \Bigg( \frac{1}{n!}\,\frac{\omega_{n,2n+2}}{2} &+ \sum_{k=1}^{n} \frac{(2n)!}{n!\,(2k)!(n-k)!\,2^{n-k}}\,\omega_{n,2k} \\ &+ \sum_{k=1}^{n} \frac{(2n)!}{n!\,(2k-2)!(n+1-k)!\,2^{n-k+2}}\,\omega_{n,2k}\Bigg)\end{aligned}$$

For $\omega_{n,N}$, from $\int_0^1 (1-t)t^k dt = \frac{k!}{(k+2)!}$ and Fubini's theorem, we obtain

$$\begin{aligned}\omega_{n,N} &= \sum_{\substack{\gamma\in\mathbb{N}_0^{n+1}\\|\gamma|=N}} \frac{N!}{\gamma!}\int_X (1-t_1)t_1^{\gamma_1}\,|\alpha_1|^{2+\gamma_1}\cdots(1-t_n)t_n^{\gamma_n}\,|\alpha_n|^{2+\gamma_n}|\beta|^{\gamma_0}\,d|\Gamma_n|(\xi) \\ &= \sum_{\substack{\gamma\in\mathbb{N}_0^{n+1}\\|\gamma|=N}} \frac{N!(\gamma_0+2)!}{\gamma_0!(\gamma+2)!}\int_{\mathbb{R}^d\times(\mathbb{R}^d)^n} |\alpha_1|^{2+\gamma_1}\cdots|\alpha_n|^{2+\gamma_n}|\beta|^{\gamma_0}\,d(|\nu_d|\otimes|\mu_d|^{\otimes n})(\beta,\alpha_1,\dots,\alpha_n) \\ &\overset{(3.5)}{\leqslant} C_\nu\,C_\mu^n \sum_{|\gamma|=N} \frac{N!(\gamma_0+2)!}{\gamma_0!(\gamma+2)!}\,\frac{\gamma_0!}{\varepsilon^{\gamma_0}}\,\frac{(\gamma_1+2)!\cdots(\gamma_n+2)!}{\varepsilon^{2+\gamma_1}\cdots\varepsilon^{2+\gamma_n}} = \frac{N!}{\varepsilon^{2n+N}}\,C_\nu\,C_\mu^n \sum_{|\gamma|=N} 1 \\ &= \frac{(N+n)!}{\varepsilon^{2n+N}n!}\,C_\nu\,C_\mu^n\end{aligned}$$

Hence

$$\begin{aligned}|J_n(h)-J_n(0)| \leqslant |h|\; C_\nu C_\mu^n \Bigg( \frac{(3n+2)!}{2\,\varepsilon^{4n+2}n!\,n!} &+ \sum_{k=1}^{n} \frac{(2n)!}{(2k)!(n-k)!\,2^{n-k}}\,\frac{(n+2k)!}{\varepsilon^{2(n+k)}n!\,n!} \\ &+ \sum_{k=1}^{n} \frac{(2n)!}{(2k-2)!(n+1-k)!\,2^{n-k+2}}\;\frac{(n+2k)!}{\varepsilon^{2(n+k)}n!\,n!}\Bigg)\end{aligned} \qquad (iii)$$

Next, for any $x\in\mathcal{H}$, let us write

$$\begin{aligned}V_a(x)-V(a) &= \int_{\mathcal{H}} (e^{i\langle x,\alpha\rangle}-1)\,d\mu^a(\alpha) = \int_{\mathcal{H}}\int_0^1 \frac{d}{dt}e^{it\langle x,\alpha\rangle}\,dt\,d\mu^a(\alpha) \\ &= i\int_{\mathcal{H}}\int_0^1 \langle x,\alpha\rangle\,e^{it\langle x,\alpha\rangle}\,dt\,d\mu^a(\alpha)\end{aligned}$$

As above, we obtain for any $x\in\mathbb{R}^d$

$$w(x) = i\int_{\mathbb{R}^d}\int_0^1 \langle x,y\rangle e^{it\langle x,y\rangle}\,dt\,d\mu_d(y) - \langle x,b\rangle$$

where $b := \gamma_E(\mathcal{P}_n a)$. Then, if $Y_k := \mathbb{R}^d \times ([0,1]\times\mathbb{R}^d)^k$, by Fubini's theorem

$$\varphi_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} (i)^k (-\langle x, b\rangle)^{n-k} \int_{Y_k} \langle x,\alpha_1\rangle \cdots \langle x,\alpha_k\rangle \, e^{i\langle x, f_k(\zeta)\rangle}\, d\Gamma_k(\zeta)$$

where $\Gamma_k := \nu_d \otimes (\lambda_{[0,1]} \otimes \mu_d)^{\otimes k}$, $\zeta = (\beta, (t_1,\alpha_1), \ldots, (t_k,\alpha_k))$. From Lemma 11 and Proposition 13, it follows

$$\begin{aligned}
&\int^o_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} \varphi_n(x)\,dx \\
&= \sum_{k=0}^{n} \tbinom{n}{k} (i)^k (-1)^{n-k} \int_{Y_k} \int^o_{\mathbb{R}^d} e^{\frac{i}{2h}|x|^2} \langle x,\alpha_1\rangle \cdots \langle x,\alpha_k\rangle \, \langle x,b\rangle^{n-k}\, e^{i\langle x, f_k(\zeta)\rangle}\, dx\, d\Gamma_k(\zeta) \\
&= \sum_{k=0}^{n} \tbinom{n}{k} (2\pi i h)^{d/2} (-1)^{n-k} \int_{Y_k} e^{-\frac{ih}{2}|f_k(\zeta)|^2} H_n(\alpha_1,\ldots,\alpha_k, b, \ldots, b|h)(f_k(\zeta))\, d\Gamma_k(\zeta) \\
&= (2\pi i h)^{d/2} \sum_{k=0}^{n} \tbinom{n}{k} (-1)^{n-k} \int_{Y_k} e^{-\frac{ih}{2}|f_k(\zeta)|^2} \sum_{m=0}^{\lfloor n/2\rfloor} (-ih)^{n-m}\, \eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))\, d\Gamma_k(\zeta)
\end{aligned}$$

where

$$|\eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))| \;\leqslant\; \frac{n!}{(n-2m)!\, m!\, 2^m} |\alpha_1| \cdots |\alpha_k|\, |b|^{n-k}\, |f_k(\zeta)|^{n-2m} \qquad (iv)$$

Therefore

$$\begin{aligned}
J_n(h) \;&=\; \frac{1}{n!} \sum_{k=0}^{n} \tbinom{n}{k} (-1)^k \int_{Y_k} e^{-\frac{ih}{2}|f_k(\zeta)|^2} \sum_{m=0}^{\lfloor n/2\rfloor} (i/h)^m\, \eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))\, d\Gamma_k(\zeta) \\
&= \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{k=0}^{n} \frac{(-1)^k (i/h)^m}{k!(n-k)!} \int_{Y_k} e^{-\frac{ih}{2}|f_k(\zeta)|^2} \eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))\, d\Gamma_k(\zeta) \\
&= \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{k=0}^{n} \sum_{l=0}^{m} \frac{(-1)^k (-ih)^{l-m}}{k!(n-k)!\, 2^l\, l!} \int_{Y_k} |f_k(\zeta)|^{2l}\, \eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))\, d\Gamma_k(\zeta) \\
&+ \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \int_{Y_k} e_{m+1}\Big(\frac{h}{2}|f_k(\zeta)|^2\Big)\, \eta^m_{\alpha_1,\ldots,\alpha_k,b,\ldots,b}(f_k(\zeta))\, d\Gamma_k(\zeta)
\end{aligned}$$

Thus, for $n\in\mathbb{N}$, $m\leqslant\lfloor n/2\rfloor$, $s\leqslant m$, there are constants $b_{n,m,s} \in \mathbb{C}$ such that

$$\left| J_n(h) - \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{s=0}^{m} b_{n,m,s}\, h^{-s} \right| \;\leqslant\; \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{\lfloor n/2\rfloor} \frac{|h|^{m+1}\, |b|^{n-k}}{2^{2m+1}(n-2m)!\, m!\, (m+1)!}\, \tilde{\omega}_{n,k}$$

where we have used $(iv)$, and we have set

$$
\begin{aligned}
\tilde\omega_{n,k} &:= \int_{Y_k} |f_k(\zeta)|^{n+2}|\alpha_1|\cdots|\alpha_k|\, d|\mathit{\Gamma}_k|(\zeta) \\
&\leqslant \sum_{\substack{\gamma\in\mathbb{N}_0^{k+1}\\ |\gamma|=n+2}} \frac{(n+2)!}{\gamma!}\int_{Y_k} t_1^{\gamma_1}\,|\alpha_1|^{1+\gamma_1}\cdots t_n^{\gamma_k}\,|\alpha_k|^{1+\gamma_k}|\beta|^{\gamma_0}\, d|\mathit{\Gamma}_k|(\xi) \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^{k+1}\\ |\gamma|=n+2}} \frac{(n+2)!(\gamma_0+1)}{(\gamma+1)!}\int_{\mathbb{R}^d\times(\mathbb{R}^d)^n} |\alpha_1|^{1+\gamma_1}\cdots|\alpha_k|^{1+\gamma_k}|\alpha_0|^{\gamma_0}\, d\big(|\nu_d|\otimes|\mu_d|^{\otimes k}\big)(\alpha_0,\ldots,\alpha_k) \\
&\overset{(3.5)}{\leqslant} C_\nu C_\mu^k \sum_{|\gamma|=n+2}\frac{(n+2)!\,(\gamma_0+1)}{(\gamma+1)!}\frac{(\gamma_1+1)!}{\varepsilon^{\gamma_1+1}}\cdots\frac{(\gamma_k+1)!}{\varepsilon^{\gamma_k+1}}\frac{\gamma_0!}{\varepsilon^{\gamma_0}} = C_\nu C_\mu^k\frac{(n+2)!}{\varepsilon^{n+2+k}}\sum_{|\gamma|=n+2} 1 \\
&= C_\nu C_\mu^k \frac{(n+2+k)!}{k!\,\varepsilon^{n+k+2}}
\end{aligned}
$$

Hence, together with

$$|b| \leqslant \|a\| \leqslant \int_{\mathcal{H}} \|x\| d|\mu|(x) \leqslant \frac{C_\mu}{\varepsilon}$$

we find that for $n\in\mathbb{N}$, $r\leqslant\lfloor n/2\rfloor$, there are constants $c_{n,r}\in\mathbb{C}$, such that for $|h|<1$,

$$\Big|J_n(h) - \sum_{r=0}^{\lfloor n/2\rfloor} c_{n,r}\,h^{-r}\Big| \leqslant |h|\,\frac{C_\nu}{\varepsilon^2}\Big(\frac{C_\mu}{\varepsilon^2}\Big)^n \sum_{k=0}^{n}\sum_{m=0}^{\lfloor n/2\rfloor}\frac{n!\,(n+2+k)!}{2^{2m+1}(n-2m)!\,m!\,(m+1)!\,k!\,k!\,(n-k)!}$$

Comparing with $(iii)$, we obtain that $c_{n,r}=0$ for all $r=1,\ldots,\lfloor n/2\rfloor$, $n\in\mathbb{N}$, i.e.

$$\Big|J_n(h) - c_{n,0}\Big| \leqslant |h|\,\frac{C_\nu}{2\varepsilon^2}\Big(\frac{C_\mu}{2\varepsilon^2}\Big)^n n!\sum_{k=0}^{n}\frac{(n+2+k)!}{k!\,k!\,(n-k)!}\sum_{m=0}^{\lfloor n/2\rfloor}\frac{2^{n-2m}}{(n-2m)!\,m!\,(m+1)!}$$

From $\binom{2m}{m}\leqslant 2^{2m}$, it follows

$$\sum_{m=0}^{\lfloor n/2\rfloor}\frac{n!\,2^{-2m}}{(n-2m)!\,m!\,(m+1)!} \leqslant \sum_{m=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2m)!\,(2m)!} \leqslant \sum_{k=0}^{n}\binom{n}{k} = 2^n$$

Moreover, by [30, 3a]

$$\binom{n+2+k}{k} = \sum_{k=0}^{n}\binom{n+2}{l}\binom{k}{k-l}$$

Therefore,

$$\sum_{k=0}^{n} \frac{(n+2+k)!}{k!\,k!\,(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+2+k)!}{n!k!} = \frac{(n+2)!}{n!} \sum_{k=0}^{n} \binom{n+2+k}{k}\binom{n}{k}$$

$$= \frac{(n+2)!}{n!} \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{n+2}{l}\binom{k}{k-l}\binom{n}{k} \leqslant 2^{n+2}\,\frac{(n+2)!}{n!} \sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}$$

$$= 2^{n+2}\,\frac{(n+2)!}{n!}\sum_{k=0}^{n}\binom{n}{k}2^k = 2^{n+2}\,\frac{(n+2)!}{n!}\,3^n = 2^2\,6^n\,\frac{(n+2)!}{n!}$$

Thus, for all $n \in \mathbb{N}$ and $h \in \mathbb{R}$ with $|h| < 1$, we obtain

$$\Big|J_n(h) - c_{n,0}\Big| \leqslant |h|\; 2\,\frac{C_\nu}{\varepsilon^2}\left(12\,\frac{C_\mu}{\varepsilon^2}\right)^n \frac{(n+2)!}{n!} = |h|\;\frac{2C_\nu}{\varepsilon^2}\;\frac{(n+2)!}{n!}\;\gamma^n$$

where $\gamma := 12\,\frac{C_\mu}{\varepsilon^2} < 1$, by assumption. If for each $N \in \mathbb{N}$, set $S_N(h) := \sum_{n=0}^{N} J_n(h)$ and $c_N := \sum_{n=0}^{N} c_{n,0}$, then from $\gamma < 1$ it follows

$$\Big|S_N(h) - c_N\Big| \leqslant |h|\,\frac{2C_\nu}{\varepsilon^2}\sum_{n=0}^{N}(n+2)(n+1)\,\gamma^n \leqslant |h|\,\frac{2C_\nu}{\varepsilon^2}\,(1-\gamma)^{-3}$$

Since $J(h) = I_q(h) = \lim_{N\to\infty} S_N(h)$, it follows from Lemma 8, that $(S_N(0))_{N\in\mathbb{N}}$ is convergent, and moreover, that

$$\big|I_q(h) - I_q(0)\big| \leqslant C\,|h|$$

for all $h \in \mathbb{R}$ with $|h|<1$, where $C := \frac{2C_\nu}{\varepsilon^2}\left(1-12\,\frac{C_\mu}{\varepsilon^2}\right)^{-3}>0$, and $I_q(0) := \lim_{N\to\infty} S_N(0)$. Since $C$ does not depend on $d$ or $q$, we may use Lemma 8 again, in order to obtain

$$\big|I^*(h) - I^*(0)\big| \leqslant C\,|h| \qquad (v)$$

where $I^*(0) := \lim_{q\to\infty} I_q(0)$. From above and the proof of Lemma 10,

$$J_n(0) = \frac{(-1)^n}{n!}\int_X (1-t_1)\cdots(1-t_n)\,\gamma^n_{\alpha_1,\ldots,\alpha_n}(f_n(\xi))\,d\varGamma_n(\xi)$$

$$= \frac{(-1)^n\hat{\nu}_d(0)}{n!\,n!\,2^n}\sum_{\sigma\in S}\int_{(\mathbb{R}^d)^n}\langle\alpha_{\sigma(1)},\alpha_{\sigma(2)}\rangle\cdots\langle\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)}\rangle\,d\mu_d^{\otimes n}(\alpha_1,\ldots,\alpha_n)$$

where $S$ denotes the set of all permutations of $(1,1,\ldots,n,n)$, which consists of $(2n)!\,2^{-n}$ elements. Moreover, if we denote $D^2V(a)$ by $T$, considered as a map from $\mathcal{H}$ to $\mathcal{H}$ by using Riesz representation theorem, i.e. for any $x, y \in \mathcal{H}$,

$$\langle x, Ty\rangle = -\int_{\mathcal{H}}\langle x,\alpha\rangle\langle\alpha,y\rangle\,d\mu^a(\alpha)$$

from Proposition 11. Since $\mu_d = \mu^a \circ \mathcal{P}_q^{-1} \circ \gamma_E^{-1}$, it follows for all $x, y \in \mathbb{R}^d$,

$$-\int_{\mathbb{R}^d} \langle x, \alpha \rangle \langle \alpha, y \rangle \, d\mu_d(\alpha) = \langle x, T_q y \rangle$$

where $T_q := \gamma_E \mathcal{P}_q T \mathcal{P}_q \gamma_E^{-1}$. Hence, for any $n \in \mathbb{N}$, and $x, y \in \mathbb{R}^d$,

$$\langle x, T_q^n y \rangle \;=\; (-1)^n \int_{(\mathbb{R}^d)^n} \langle x, \alpha_1 \rangle \langle \alpha_1, \alpha_2 \rangle \cdots \langle \alpha_{n-1}, \alpha_n \rangle \langle \alpha_n, y \rangle \, d\mu_d^{\otimes n}(\alpha_1, \ldots, \alpha_n)$$

And, therefore the trace of $T_q^n$ is given by

$$\sum_{j=1}^{d} \langle e_j, T_q^n e_j \rangle \;=\; (-1)^n \int_{(\mathbb{R}^d)^n} \langle \alpha_1, \alpha_2 \rangle \cdots \langle \alpha_{n-1}, \alpha_n \rangle \langle \alpha_n, \alpha_1 \rangle \, d\mu_d^{\otimes n}(\alpha_1, \ldots, \alpha_n) \qquad (vi)$$

where $\{e_j\}_{j=1}^d$ is any ONB in $\mathbb{R}^d$. Thus, it follows

$$|tr(T_q^n)| \leqslant \Big( \int_{\mathbb{R}^d} |\alpha|^2 d|\mu_d|(\alpha) \Big)^n \leqslant \Big( \int_{\mathcal{H}} \|\alpha\|^2 d|\mu|(\alpha) \Big)^n \leqslant \left( \frac{2C_\mu}{\varepsilon^2} \right)^n < 1$$

since by assumption $12C_\mu < \varepsilon^2$. Hence, the series $\sum_{n=1}^{\infty} tr(T_q^n)/n$ converges absolutely and, e.g. from [15, Theorem 3.3], we obtain

$$\det\big(I - T_q\big) \;=\; \exp\left( -\sum_{n=1}^{\infty} \frac{tr(T_q^n)}{n} \right)$$

where $I$ denotes the $d$-dimensional identity matrix, and therefore

$$\det(I - T_q)^{-\frac{1}{2}} \;=\; \exp\left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{tr(T_q^n)}{n} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, n!\, 2^n} \sum_{\sigma \in S} \int_{(\mathbb{R}^d)^n} \langle \alpha_{\sigma(1)}, \alpha_{\sigma(2)} \rangle \cdots \langle \alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)} \rangle \, d\mu_d^{\otimes n}(\alpha_1, \ldots, \alpha_n)$$

where the last equality follows from $(vi)$ by using the combinatorial result [29, 3.28]. Hence, we have found

$$I_q(0) = \sum_{n=0}^{\infty} J_n(0) = \det(I - T_q)^{-\frac{1}{2}} g(a) \qquad (vii)$$

Next, since the determinant of finite-rank operators can be calculated by means of their basis representation, i.e.

$$\det\big(\mathbb{1} - \gamma_E^{-1} T_q \gamma_E\big) \;=\; \det(I - T_q)$$

let us show that $\gamma_E^{-1} T_q \gamma_E = P_q T P_q$ converges in trace norm to $T = D^2 V(a)$, so that by Definition 7, $\mathrm{Det}(\mathbb{1} - D^2 V(a))$ is obtained as the limit of $\det(\mathbb{1} - P_q T P_q)$, as $q \to \infty$.

For any two orthonormal bases $\{e_n\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, from monotone convergence, the Cauchy-Schwarz inequality in $l^2$ and Parseval's identity, it follows

$$\begin{aligned}
&\Big|\sum_{n\in\mathbb{N}}\langle e_n,\big(\mathcal{P}_q D^2V(a)\mathcal{P}_q - D^2V(a)\big)f_n\rangle\Big| \\
&\quad\leqslant \sum_n \int_{\mathcal{H}} \big|\langle e_n,\mathcal{P}_q\alpha\rangle\langle\mathcal{P}_q\alpha,f_n\rangle - \langle e_n,\alpha\rangle\langle\alpha,f_n\rangle\big|\, d|\mu|(\alpha) \\
&\quad\leqslant \int_{\mathcal{H}}\sum_n \big|\langle e_n,\mathcal{P}_q\alpha\rangle\langle\mathcal{P}_q\alpha-\alpha,f_n\rangle\big|\, d|\mu|(\alpha) + \int_{\mathcal{H}}\sum_n \big|\langle\alpha,f_n\rangle\langle e_n,\mathcal{P}_q\alpha-\alpha\rangle\big|\, d|\mu|(\alpha) \\
&\quad\leqslant \int_{\mathcal{H}}\Big(\sum_n|\langle e_n,\mathcal{P}_q\alpha\rangle|^2\Big)^{1/2}\Big(\sum_n|\langle f_n,\mathcal{P}_q\alpha-\alpha\rangle|^2\Big)^{1/2} d|\mu|(\alpha) \\
&\qquad\qquad + \int_{\mathcal{H}}\Big(\sum_n|\langle f_n,\alpha\rangle|^2\Big)^{1/2}\Big(\sum_n|\langle e_n,\mathcal{P}_q\alpha-\alpha\rangle|^2\Big)^{1/2} d|\mu|(\alpha) \\
&\quad= 2\int_{\mathcal{H}}\|\alpha\|\,\|\mathcal{P}_q\alpha-\alpha\|\, d|\mu|(\alpha) \xrightarrow{q\to\infty} 0
\end{aligned}$$

by dominated convergence, and $\mathcal{P}_q$ converging strongly to the identity. Since the convergence is uniform in $\{e_n\}_n$ and $\{f_n\}_n$, from (3.4), we obtain that $\mathcal{P}_q D^2V(a)\mathcal{P}_q$ converges to $D^2V(a)$ in trace norm. Hence $(v)$ together with $(vii)$ prove the claim. □

## 3.2 Application to the Feynman-Fresnel path integral

In this section, we apply the stationary phase approximation of Theorem 4 to the path integral solution for the Schrödinger equation found in Theorem 3, in order to find an approximation of such solutions in the semiclassical regime.

In the semiclassical theory of quantum mechanics, one considers the data of the system to live on a much larger scale than the quantum wavelength. This can be described by introducing a small parameter $h>0$, and rescaling the potential $V$ and the initial datum $\varphi$ by $\tilde{V}(x) := V(hx)$ and $\tilde{\varphi}(x) := \varphi(hx)$. Passing from microscopic $(x,t)$ to macroscopic coordinates $(hx,ht) =: (\xi,\tau)$, then Schrödinger's equation (2.30) reads

$$ih\frac{d\psi^h_\tau}{d\tau} = -\frac{h^2}{2}\Delta_\xi\psi^h_\tau + V\psi^h_\tau \tag{3.15}$$

where $\psi^h_\tau(\xi) := \psi_{\frac{\tau}{h}}\big(\frac{\xi}{h}\big)$, and $\psi_t$ denotes a solution to (2.30) with potential $\tilde{V}$ and initial datum $\tilde{\varphi}$. The following simple result shows how this rescaling applies to Fresnel integrals.

**Lemma 14** *(Scaling property). For some $t>0$ and $h>0$, let $f=\hat{\mu}_f \in \mathscr{F}(\mathcal{H}_t)$, and let $f_h$ denote the function on $\mathcal{H}_{\frac{t}{h}}$, given by $f_h(x) := f\big(s\mapsto h\,x(\frac{s}{h})\big)$, then it holds*

$$\mathcal{F}_{\frac{t}{h}}(f_h) = \mathcal{F}^h_t(f)$$

***Proof.*** First, let $\mathcal{S}_h$ denote the scaling from $\mathcal{H}_t$ to $\mathcal{H}_{\frac{t}{h}}$, given by $\mathcal{S}_h(x) = \big(s \mapsto x(hs)\big)$, then we observe that

$$\Big\langle s \mapsto h\, x\Big(\frac{s}{h}\Big), y \Big\rangle_t = \int_0^t \dot{x}\Big(\frac{s}{h}\Big)\dot{y}(s)\, ds = \int_0^{t/h} \dot{x}(s)\, h\, \dot{y}(hs) ds = \langle x, \mathcal{S}_h y\rangle_{t/h}$$

and so $f_h(x) = \int_{\mathcal{H}_t} e^{i\langle x, \mathcal{S}_h y\rangle_{t/h}}\, d\mu_f(y)$ is the Fourier transform of the complex image measure $\mu_f \circ \mathcal{S}_h^{-1}$, evaluated at $x \in \mathcal{H}_{\frac{t}{h}}$. Therefore, from Theorem 2,

$$\mathcal{F}_{\frac{t}{h}}(f_h) = \int_{\mathcal{H}_{t/h}} e^{-\frac{i}{2}\|x\|^2_{t/h}}\, d\mu_f \circ \mathcal{S}_h^{-1}(x) = \int_{\mathcal{H}_t} e^{-\frac{i}{2}\|\mathcal{S}_h x\|^2_{t/h}}\, d\mu_f(x)$$

Now, a simple calculation shows that $\|\mathcal{S}_h(x)\|^2_{t/h} = h\,\|x\|^2_t$. Hence, another application of Theorem 2 gives

$$\int_{\mathcal{H}_t} e^{-\frac{i}{2}\|\mathcal{S}_h x\|^2_{t/h}}\, d\mu_f(x) \;=\; \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2h}\|x\|^2_t} f(x) dx$$

which, by definition of $\mathcal{F}_t^h$, proves the desired result, $\mathcal{F}_{\frac{t}{h}}(f_h) = \mathcal{F}_t^h(f)$. □

This shows, how $\psi_\tau^h$ can be constructed, if the conditions of Theorem 3 are satisfied, and moreover, that we can use Theorem 4 to obtain a semiclassical approximation of its FFPI representation. The statement of Theorem 5 and its proof are a direct application of Theorem 4, similar to [2, Theorem 3.7].

As was the case with Theorem 3, we have to impose quite strong conditions on the potential and the initial data. In particular, we not only need that they are in the Fresnel class $\mathscr{F}(\mathbb{R}^d)$, but moreover, assumptions (3.16) require that all moments of their underlying measures have to exist, which (similar to Proposition 11) implies that $V$ and $\varphi$ have to be in $C^\infty(\mathbb{R}^d)$, with bounded derivatives of all orders.

**Theorem 5** *(Stationary phase approximation of FFPIs). Let* $V{=}\hat{\mu}$ *and* $\varphi{=}\hat{\nu}$ *belong to* $\mathscr{F}(\mathbb{R}^d)$, *such that there are constants* $C_\mu, C_\nu, \varepsilon > 0$, *with*

$$\int_{\mathbb{R}^d} |x|^j d|\mu|(x) \;\leqslant\; C_\mu \frac{j!}{\varepsilon^j}\;, \quad \int_{\mathbb{R}^d} |x|^j d|\nu|(x) \;\leqslant\; C_\nu \frac{j!}{\varepsilon^j} \tag{3.16}$$

*for all* $j \in \mathbb{N}$, *then the assumptions of Theorem 3 are fullfilled, and therefore*

$$\psi_t^h(\xi) \;:=\; \mathcal{F}_t^h\Big(x \mapsto e^{-\frac{i}{h}\int_0^t V(x(s)+\xi) ds}\, \varphi(x(0)+\xi)\Big) \tag{3.17}$$

*forms a solution to the Schrödinger equation (3.15). If we further assume* $12tC_\mu{<}\varepsilon_t^2$, *where* $\varepsilon_t := t^{-\frac{1}{2}}\varepsilon$, *then for each* $\xi \in \mathbb{R}^d$, *the phase function* $\Phi_\xi$, *given by* $\Phi_\xi(x) = \frac{1}{2}\|x\|_t^2 - \int_0^t V(x(s){+}\xi)\, ds$, *admits a unique stationary path* $a_\xi \in \mathcal{H}_t$, *and for* $|h|{<}1$, *it holds*

$$\psi_t^h(\xi) \;=\; \mathrm{Det}(D^2\Phi_\xi(a_\xi))^{-\frac{1}{2}}\, e^{\frac{i}{h}\Phi(a_\xi)} \varphi(a_\xi(0){+}\xi) + R(h) \tag{3.18}$$

*where* $R(h) \to 0$, *as* $h \to 0$.

***Proof.*** From Proposition 11, and (3.16), it follows $\partial^\alpha f(x) = i^{|\alpha|}\int_{\mathbb{R}^d} y^\alpha\, e^{i\langle x,y\rangle}\, d\mu_f(y)$, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant 2$, where $f \in \{V,\varphi\}$, and so the assumptions of Theorem 3 are satisfied. If we denote the argument of the Fresnel integral (3.17) by $u$, i.e. $\psi_t^h(\xi) = \mathcal{F}_t^h(u)$, as well as $u_h(x) := u\big(s \mapsto h\, x(\frac{s}{h})\big)$, then a simple change of variables yields

$$u_h(x) \;=\; e^{-\frac{i}{h}\int_0^t V(hx(s/h)+\xi)ds}\varphi(h\,x(0){+}\xi) \;=\; e^{-i\int_0^{t/h}\tilde{V}(x(s)+\xi/h)ds}\tilde{\varphi}(x(0){+}\xi/h)$$

where $\tilde{V}(x) := V(hx)$ and $\tilde{\varphi}(x) := \varphi(hx)$. Hence, from the scaling property (Lemma 14), it follows $\psi_t^h(\xi) = \psi_{\frac{t}{h}}(\frac{\xi}{h})$, where $\psi_t$ denotes the FFPI defined in Theorem 3, forming a solution to the Schrödinger equation (2.30) with potential $\tilde{V}$ and initial datum $\tilde{\varphi}$. Thus, by Theorem 3 and the discussion around equation (3.15), $\psi_t^h$ forms a solution to (3.15). Next, if $\nu_0, \mu_t$ denote the complex measures in $\mathscr{F}(\mathcal{H}_t)$ with $\hat{\nu}_0(x) = \varphi(x(0){+}\xi)$ and $\hat{\mu}_t = \int_0^t V(x(s){+}\xi)ds$, and $(\gamma_s^i)_{i=1}^d \subset \mathcal{H}_t$ are the paths defined in the proof of Proposition 4 satisfying $\langle x, \gamma_s^i\rangle_t = x_i(s)$, then from the construction in Lemma 5 and the proof of Proposition 4,

$$\int_{\mathcal{H}_t} \|x\|^j d|\nu_0|(x) \;=\; \int_{\mathbb{R}^d} \Big\|\sum_{i=1}^d v_i\gamma_0^i\Big\|_t^j d|\nu|(v) \;\leqslant\; t^{j/2}\int_{\mathbb{R}^d} |v|^j d|\nu|(v) \;\leqslant\; C_\nu \frac{j!}{(t^{-\frac{1}{2}}\varepsilon)^j}$$

due to $\|\gamma_0^j\|_t = t^{1/2}$, and also

$$\begin{aligned}\int_{\mathcal{H}_t} \|x\|^j d|\mu_t|(x) \;&=\; \int_0^t\int_{\mathbb{R}^d} \Big\|\sum_{i=1}^d v_i\gamma_s^i\Big\|_t^j d|\mu|(v)ds \;\leqslant\; \int_0^t (t{-}s)^{j/2}\int_{\mathbb{R}^d} |v|^j d|\mu|(v)\\ &=\; \frac{t^{j/2+1}}{\frac{j}{2}{+}1}\int_{\mathbb{R}^d} |v|^j d|\mu|(v) \;\leqslant\; t\,C_\mu \frac{j!}{(t^{-\frac{1}{2}}\varepsilon)^j} \;=:\; C_{\mu,t}\frac{j!}{\varepsilon_t^j}\end{aligned}$$

Therefore, the assumptions of Theorem 4 are fullfilled, i.e. there for each $\xi \in \mathbb{R}^d$, there exists a unique stationary point $a_\xi$ of $\Phi_\xi$, and (3.18) follows from (3.6). □

As we can see from Theorem 5, as $h \to 0$, the main contribution to the Feynman-Fresnel path integral $\psi_t^h(\xi)$ comes from the stationary path $a_\xi$, in the sense that, at lowest order in $h$, the information carried by $\psi_t^h(\xi)$ is determined only by $a_\xi$. In order to find out more on $a_\xi$, let us first calculate the Fréchet derivative of $\Phi_\xi$.

**Proposition 15**. *For any $\xi \in \mathbb{R}^d$, the Fréchet derivative of the phase function $\Phi_\xi$, defined on $\mathcal{H}_t$ by $\Phi_\xi(x) := \frac{1}{2}\|x\|_t^2 - \int_0^t V(x(s){+}\xi)ds$, is given by*

$$\big(D\Phi_\xi(x)\big)(y) \;=\; \langle x, y\rangle_t - \int_0^t y(s)\cdot\nabla V(x(s){+}\xi)\,ds \tag{3.19}$$

*for all $y \in \mathcal{H}_t$.*

***Proof.*** The Fréchet derivative of $x \mapsto \|x\|_t^2$ is $2\langle x, \cdot\rangle_t$, as is true on any Hilbert space. For $U(x) := \int_0^t V_\xi(x(s))ds$, where for any $\alpha \in \mathbb{R}^d$, $V_\xi(\alpha) := V(\alpha{+}\xi)$, by Taylor's theorem we may write

$$V_\xi(x(s)+h(s)) \;=\; V_\xi(x(s)) + h(s)\cdot\nabla V_\xi(x(s)) + R(x(s),h(s))$$

where $|R(x(s),h(s))| \leqslant C\,|h(s)|^2$, for all $x,h \in \mathcal{H}_t$, $s\in[0,t]$. Therefore

$$\Big|U(x+h) - U(x) - \int_0^t h(s)\cdot\nabla V_\xi(x(s))\,ds\Big| \;\leqslant\; \int_0^t |R(x(s),h(s))|\,ds \;\leqslant\; C\,\|h\|_2^2$$

Since $h\in H^1([0,1])$, we have for any $a,b\in[0,t]$,

$$\int_a^b h(s)h'(s)ds \;=\; h(b)^2 - h(a)^2 - \int_a^b h'(s)h(s)ds$$

from integration by parts on $H^1$. Hence $|h(a)|^2 \leqslant |h(b)|^2 + 2\int_a^b |h(s)|\,|h'(s)|\,ds$, and by Cauchy-Schwarz, $\int_a^b |h||h'| \leqslant \|h\|_2\|h\|_t$. In particular, for $b{=}t$, due to $h(t){=}0$, we find

$$\|h\|_2^2 \;=\; \int_0^t |h(s)|^2 ds \;\leqslant\; t \sup_{s\in[0,t]} |h(s)|^2 \;\leqslant\; 2t\,\|h\|_2\|h\|_t$$

So for $h\neq 0$, $\|h\|_2 \leqslant 2t\,\|h\|_t$, which proves the claim. □

Having achieved an explicit expression for $D\Phi_\xi$, we easily obtain the following property of the stationary paths $a_\xi$, which was already proven in [2, Lemma 3.3].

**Proposition 16** *(Newton's equation). Under the assumptions of Theorem 5, the unique stationary path $a_\xi$ of $\Phi_\xi$ is in $H^2(0,t;\mathbb{R}^d)$, and the path given by $\gamma(s) := a_\xi(s)+\xi$ satisfies the following boundary value problem*

$$\begin{cases} \ddot\gamma(s) = -\nabla V(\gamma(s)) \\ \gamma(t) = \xi,\;\; \dot\gamma(0) = 0 \end{cases} \tag{3.20}$$

***Proof.*** From Proposition 15 and $D\Phi_\xi(a_\xi){=}0$, it follows for any $\omega\in C_0^\infty((0,t),\mathbb{R}^d)$, $\langle\dot\gamma,\dot\omega\rangle_2 = \langle\nabla V\circ\gamma,\omega\rangle_2$. Thus $\dot\gamma$ has the weak derivative $-\nabla V\circ\gamma$, and due to $C{:=}\|\nabla V\|_\infty$ being finite, we obtain $\|\ddot\gamma\|_2 \leqslant C\,t$. Hence $\ddot\gamma\in L^2(0,t)$, and $\ddot\gamma = -\nabla V\circ\gamma$. Moreover, from integration by parts in $H^1$, it follows for any $h\in\mathcal{H}_t$, with $h(0)\neq 0$,

$$\begin{aligned}\dot\gamma(0)h(0) \;&=\; \dot\gamma(t)h(t) - \int_0^t \ddot\gamma(s)h(s)\,ds - \int_0^t \dot\gamma(s)\dot h(s)\,ds \\ &=\; \int_0^t \nabla V(\gamma(s))h(s)\,ds - \int_0^t \dot\gamma(s)\dot h(s)\,ds \;=\; 0\end{aligned}$$

where we have used $(D\Phi_\xi(a_\xi))(h) = 0$ and Proposition 15. Hence $\gamma$ satisfies (3.20) □

Hence, Theorem 5 and Proposition 16 show, that at lowest order in $h$, $\psi_t(\xi)$ is determined by the classical path $\gamma = a_\xi+\xi$, with $\gamma(t)=\xi$ and $\dot\gamma(0) = 0$.

Let us add the remark, that one can also find an explicit form for the Fredholm determinant $\mathrm{Det}(D^2\Phi(a))$ from equation (3.18). In [2, Lemma 3.6], it is shown that it can be written as the Jacobian determinant of the map $\xi \mapsto a_\xi$.

Without going further into the details of semi-classical analysis, let us refer to [26], where the assertion of Theorem 5 is obtained from a completely different approach, which is connected with pseudo-differential calculus and Lagrangian analysis (see [26, Theorem 12.5]). Furthermore, there it is shown, how from (3.18) it can be rigorously derived, that *the semi-classical wave function of a quantum mechanical particle is concentrated near its classical trajectory*, a well-known fact, which is sometimes formulated as: *The semiclassical limit of quantum mechanics is given by classical mechanics.*

## 3.3 Concluding remark

In this thesis, we have studied one of the available rigorous approaches to the Feynman path integral in form of solutions to the non-relativistic Schrödinger equation. We have seen, how the oscillatory nature of path integrals can be used to find a semiclassical approximation of quantum mechanical wave functions my means of a stationary phase approximation. The overall restriction was the limited applicability to physically interesting cases of potentials, due to the condition that the potentials have to be Fourier transforms of complex measures and moreover need to satisfy certain regularity assumptions.

This is probably the reason, why this approach to Feynman path integrals wasn't able to get more attention in all the years since it was introduced by Albeverio and Hoegh-Krohn in [3]. Even though the availability of a stationary phase approximation gives a certain amount of practical potential, the theory misses non-trivial physical applications.

The whole topic of finding mathematically rigorous formulations of Feynman path integrals has one big problem: It never evolved from the stage of theory construction to a stage focussed on solving problems.

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# List of symbols

## Eidesstattliche Erklärung

Hiermit erkläre ich, Sebastian Gottwald, an Eides statt, dass ich die vorliegende Masterarbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Die Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

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## Staturory Declaration

I herewith declare that I have completed the present thesis independently making use only of the specified literature. The thesis in this form or in any other form has not been submitted to an examination body.

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